On the Distribution of Squarefree Numbers in Arithmetic Progressions

Gottfried Wilhelm Leibniz Universität Hannover
Fakultät für Mathematik und Physik
Institut für Algebra, Zahlentheorie und Diskrete Mathematik

Author Markus Schepke MASt BSc (matriculation nº 2548160)
Address 5 Fitzwilliam Road, Flat 3, Cambridge CB2 8BN, UK
eMail markus@schepke.info
Adviser Prof Jörg Brüdern (Georg-August-Universität Göttingen)
Co-adviser Prof Stefan Wewers (Leibniz Universität Hannover)

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Abstract

This thesis will present some major results on the distribution of squarefree numbers, focusing in particular on Montgomery–Hooley style variances of squarefree numbers in arithmetic progressions. Combining the work of R. C. Vaughan [Vau05], and J. Brüdern and T. D. Wooley [BW11], we will prove an asymptotic formula for a sparse variance in which the moduli are restricted to the values of a polynomial.

Zusammenfassung

Diese Arbeit präsentiert einige wichtige Resultate in der Verteilung der quadratfreien Zahlen, wobei der Schwerpunkt insbesondere auf Varianzen der quadratfreien Zahlen in arithmetischen Progressionen im Stile von Montgomery–Hooley liegt. Indem wir die Methoden von R. C. Vaughan [Vau05], und J. Brüdern und T. D. Wooley [BW11] zusammenführen, werden wir eine asymptotische Formel für eine ausgedünnte Varianz beweisen, in der die Moduln auf die Werte eines Polynoms eingeschränkt sind.

1 Introduction

1.1 Overview

We call an integer n *squarefree* or *quadratfrei* if it is not divisible by any square, i. e., if $p^2 \nmid n$ for all primes p. The characteristic function of the squarefree numbers is $\mu(n)^2$, hence we can define

$$Q(x;k,l) := \sum_{\substack{n \le x \\ n \equiv l \ (k)}} \mu(n)^2 \tag{1}$$

as the number of squarefree integers not exceeding x congruent to l modulo k. As it turns out, the asymptotic density

$$g(k,l) := \lim_{x \to \infty} \frac{Q(x;k,l)}{x} \tag{2}$$

is positive, unless (k,l) is not squarefree, in which case it obviously vanishes. We will have a closer look at $g(k,l)$ in Section 1.3. From (2) we obtain the asymptotic

$$Q(x;k,l) = g(k,l)x + E(x;k,l), \tag{3}$$

where obviously

$$E(x;k,l) = o(x).$$

We will see that indeed we have

$$E(x;k,l) \ll x^{1/2}. \tag{4}$$

The Dirichlet series associated with the characteristic function of the squarefree numbers is

$$\sum_{n=1}^{\infty} \mu(n)^2 n^{-s} = \prod_p \left(1 + p^{-s}\right) = \prod_p \left(\frac{1 - p^{-2s}}{1 - p^{-s}}\right) = \frac{\zeta(s)}{\zeta(2s)}, \tag{5}$$

where the product runs over all primes p. Under the assumption of the Riemann hypothesis we would thus expect the error to be

$$E(x;k,l) \ll (x/k)^{1/4+\varepsilon}. \tag{6}$$

We will go into some more details on the ζ-function and the role of the Riemann hypothesis in Section 1.4. However, even without this strong assumption, we can obtain results of similar strength if we consider the mean over the residue classes. For that purpose we define the variance $V(x,y)$ by

$$V(x,y) := \sum_{k \le y} \sum_{l=1}^{k} E(x;k,l)^2. \tag{7}$$

Plugging the expected error (6) into this definition yields an expected upper bound of

$$V(x,y) \ll x^{1/2+\varepsilon} y^{3/2-\varepsilon}.$$

This turns out to be the right magnitude. R. C. Vaughan [Vau05] has proved unconditionally in the more general case of k-free numbers the asymptotic

$$\begin{aligned} V(x,y) = c_1 x^{1/2} y^{3/2} &+ O\left(x^{1/4} y^{7/4} \exp\left(-c_2 \frac{(\log 2x/y)^{3/5}}{(\log\log 3x/y)^{1/5}}\right)\right) \\ &+ O\left(x^{3/2} \exp\left(-c_3 \frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right) \end{aligned}$$

for $y \leq x$, where c_1, c_2, and c_3 are certain constants. The aim of this paper is to generalise this result to a sparse variance where the moduli only take values of a polynomial, albeit less thorough bounds on the error terms.

1.2 Historic Development

Before we go into further details, we want to take a closer look at the genesis of these kinds of average results. Squarefree numbers are obviously closely related to prime numbers, but have naturally a positive asymptotic density and a more even distribution. However, one can usually obtain similar results for prime and squarefree numbers with resembling methods.

The idea of examining the square mean of error terms dates back to M. B. Barban [Bar63, Bar64, Bar66], and H. Davenport and H. Halberstam [DH66, DH68]. They introduced the definition

$$\widetilde{V}(x,y) := \sum_{k \leq y} \sum_{\substack{l=1 \\ (l,k)=1}}^{k} \left(\vartheta(x;k,l) - \frac{x}{\varphi(k)}\right)^2,$$

where $\vartheta(x;k,l)$ is Chebychev's function in arithmetic progressions, and proved an upper bound. Shortly afterwards, this was improved by H. L. Montgomery [Mon70, Mon71] and C. Hooley [Hoo75a] to an asymptotic formula which takes the shape

$$\widetilde{V}(x,y) = xy\log y + c_4 xy + O\left(x^{1/2} y^{3/2}\right) + O\left(x^2 (\log x)^{-A}\right),$$

where c_4 is a certain constant, $y \leq x$, and $A > 0$ is arbitrary.

As the distribution of the squarefree numbers is generally less erratic, the corresponding results are slightly stronger. The variance $V(x,y)$ defined in (7) has been studied by R. C. Orr [Orr69, Orr71], M. J. Croft [Cro75], and R. Warlimont [War69, War72, War80]. The latter proved the asymptotic

$$V(x,y) = c_5 x^{1/2} y^{3/2} + O\left(x^{1/4} y^{7/4} \exp\left(-c_6 (\log x/y)^{1/5}\right)\right) + O\left(x^{3/2} (\log x)^{7/2}\right),$$

where c_5 and c_6 are certain constants.

Squarefree numbers can be generalised to k-free numbers that are not divisible by any k^{th} power. Usually, results on k-free numbers are not harder to obtain than in the squarefree case. Consequently, a number of papers on the distribution of k-free numbers has been published, generalising the squarefree results. J. Brüdern, A. Granville, A. Perelli, R. C. Vaughan, and T. D. Wooley [BGP+98] have given bounds on sums over k-free numbers. Amongst other results, they proved the bound [BGP+98, Lemma 2.2],

$$V(x,y) \ll \begin{cases} x^{1+\varepsilon} y, & y \le x, \\ y^2 \log(2y), & y > x. \end{cases} \tag{8}$$

R. C. Vaughan [Vau98a, Vau98b] has examined the distribution of general sequences with similar properties to k-free numbers, and applied these results to k-free numbers in the above mentioned paper [Vau05].

One may think of several variations in the definition of the variance. P. D. T. A. Elliott [Ell01, Ell02] introduced the idea of restricting the moduli to the value of a polynomial. He gave an upper bound, which was then improved by H. Mikawa and T. P. Peneva [MP05]. J. Brüdern and T. D. Wooley [BW11] modified this theme, using the definition

$$\widetilde{V}_f(x,y) := \sum_{k \le y} f'(k) \sum_{\substack{l=1 \\ (l,f(k))=1}}^{f(k)} \left(\vartheta(x; f(k), l) - \frac{x}{\varphi(f(k))}\right)^2. \tag{9}$$

Note that the factor $f'(k)$ has been introduced to give the sparse innermost sum the necessary weight to contribute sufficiently to the variance. Their result then assures for $f(y) \le x$ that

$$\widetilde{V}_f(x,y) = x f(y) \log f(y) + c_7 x f(y) + O\left(x^{1/2} f(y)^{3/2}\right) + O\left(x^2 (\log x)^{-A}\right),$$

where $A \ge 1$ is a fixed real number and c_7 is yet another constant depending on f only.

We will combine their methods with those of Vaughan in order to prove an asymptotic similar to that above for squarefree numbers.

1.3 Remarks on the Distribution of Squarefree Numbers

In contrast to the prime number theorem, the asymptotic (3) for the number of squarefree integers not exceeding x is a straightforward and elementary calculation. We start from the identity

$$\mu(n)^2 = \sum_{m^2 \mid n} \mu(m), \tag{10}$$

which follows from $\mu(m)$ being multiplicative. We plug this into the definition (1) of $Q(x;k,l)$, exchange the order of summation, and obtain

$$\begin{aligned} Q(x;k,l) &= \sum_{\substack{n \le x \\ n \equiv l \ (k)}} \sum_{m^2 \mid n} \mu(m) \\ &= \sum_{m \le x^{1/2}} \mu(m) \sum_{\substack{n' \le x/m^2 \\ n'm^2 \equiv l \ (k)}} 1 \\ &= \sum_{\substack{m \le x^{1/2} \\ (k,m^2) \mid l}} \mu(m) \left(\frac{x(m^2,k)}{m^2 k} + O\left(1\right) \right). \end{aligned}$$

The error term here is obviously $O\left(x^{1/2}\right)$. Extending the summation over m to all positive integers yields

$$\sum_{\substack{m \le x^{1/2} \\ (k,m^2) \mid l}} \frac{\mu(m)(m^2,k)}{m^2 k} = g(k,l) + O\left(x^{-1/2}\right),$$

where

$$g(k,l) = \sum_{\substack{m=1 \\ (m^2,k) \mid l}}^{\infty} \frac{\mu(m)(m^2,k)}{m^2 k}. \tag{11}$$

We obtain the claimed formula

$$Q(x;k,l) = g(k,l)x + O\left(x^{1/2}\right).$$

Write for short

$$Q(x) := Q(x;1,1).$$

Note that

$$g(1,1) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = \zeta(2)^{-1} = \frac{6}{\pi^2},$$

which relates closely to the classic result that the probability of two random integer being coprime is $6/\pi^2$.

For future reference we will prove a stronger error bound in the special case $k = l$.

Lemma 1.1 ([Vau05, Lemma 2.2]) *Let k be an integer with $k \le x$. Then*

$$Q(x;k,k) = g(k,k)x + O\left(\sigma_0(k)(x/k)^{1/2}\right).$$

Proof. Note first that the sum on the left is 0 if k is not squarefree, as is the main term on the right. So assume that k is squarefree. Using (10) again, we obtain

$$\begin{aligned}Q(x;k,k) &= \sum_{n\le x/k} \mu(nk)^2 \\ &= \sum_{n\le x/k}\sum_{m^2|nk} \mu(m) \\ &= \sum_{\substack{m,l \\ m^2 l \le x(m^2,k)/k}} \mu(m).\end{aligned}$$

Note that $(m^2, k) = (m, k)$ as k is squarefree. We now sort according to the values of $d = (m, k)$. This is again squarefree. We then conclude

$$\begin{aligned}Q(x;k,k) &= \sum_{d|k} \sum_{\substack{r^2 l\le x/dk \\ (r^2 d, k/d)=1}} \mu(rd) \\ &= \sum_{d|k} \mu(d) \sum_{\substack{r^2 l\le x/dk \\ (r,k)=1}} \mu(r).\end{aligned}$$

We can then approximate the sum, and arrive at

$$\begin{aligned}Q(x;k,k) &= \sum_{\substack{r\le (x/dk)^{1/2} \\ (r,k)=1}} \mu(r) \left\lfloor \frac{x}{r^2 dk} \right\rfloor \\ &= x \sum_{\substack{r=1 \\ (r,k)=1}}^{\infty} \frac{\mu(r)}{r^2 dk} + O\left((x/dk)^{1/2}\right) \\ &= g(k,k)x + O\left(\sigma_0(k)(x/k)^{1/2}\right).\end{aligned}$$

□

More properties of $g(k, l)$ will be presented in Lemmata 3.1 to 3.3. A more comprehensive survey on the distribution of k-free numbers was given by F. Pappalardi [Pap05].

1.4 Results on the ζ-function

At the very core of analytic number theory we find Dirichlet series, in particular Riemann's ζ-function. This is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$$

for $\Re s > 1$. The connection to prime numbers becomes obvious by the product representation which dates back to L. EULER, and takes the shape

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

B. RIEMANN introduced in his influential paper from 1859 the idea of considering $\zeta(s)$ on the complex plane, opening a whole new branch of number theory.

It is a standard result (cf. [Tit86, Thm. 2.1]) that $\zeta(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ with the only single pole at $s = 1$ which has residue 1. This enables us to apply Perron's formula. For that, let

$$L(a, s) := \sum_{n=1}^{\infty} a(n) n^{-s}$$

be the Dirichlet series associated with the sequence $a(n)$, which we assume to be absolutely convergent for $\Re s > \sigma_0$. Then Perron's formula assures that

$$\sum_{n \le x}^{\star} a(n) = \frac{1}{2\pi i} \int_{(\vartheta)} L(a, s) x^s \frac{\mathrm{d}s}{s},$$

where the star at the sum indicates that the last term needs to be modified if x is an integer, and $\vartheta \ge \sigma_0$. We will use a modified version of this formula in Lemma 6.1.

It turns out that for prime counting purposes, the best suitable function is the von Mangoldt function

$$\Lambda(n) := \begin{cases} \log p, & n = p^k, \\ 0, & \text{otherwise.} \end{cases}$$

The associated Dirichlet series is

$$L(\Lambda, s) = -\frac{\zeta'(s)}{\zeta(s)}.$$

On applying Perron's formula, we want to move the line of integration to the left, picking up residues of the Dirichlet series in that area. The ζ-function is zero-free in the half-plane $\Re s > 1$ by the Euler product, but is known to have infinitely many zeros with $0 < \Re s < 1$. We will later make use of the fact that $\zeta(s)$ is zero-free for $\Re s \ge 1 - \varepsilon$ (cf. [Tit86, Thm. 3.8]). This indeed is equivalent to the Prime Number Theorem, and was first proved independently by J. HADAMARD [Had96] and C. DE LA VALLÉE-POUSSIN [dlVP96].

As we see, each of these zeros constitutes a pole of $L(\Lambda, s)$, thus adding a residue to the calculation. It is for this reason that we are interested in the distribution of the zeros of $\zeta(s)$. Riemann conjectured in his paper that all zeros in the critical strip $0 \le \Re s \le 1$ have $\Re s = \frac{1}{2}$, thus implying that the primes are "as evenly distributed as possible".

Regarding the Dirichlet series (5) associated to the characteristic function of the squarefree numbers we see why we can expect the error bound (6) for $Q(x; k, l)$ under the assumption of the Riemann hypothesis. However, over 150 years later, this still remains unproven. Therefore we need to look at average results in order to gain the same strength.

The statement of our main theorem and an outline of the proof are presented in the next chapter.

2 Outline of the Proof

The purpose of this chapter is to give an overview of the main ingredients of the proof, along with a sketch of the methods to be applied. It also contains some of the most important definitions and thus serves as a reference. A comprehensive summary of the notation used along with further remarks is given at page 53.

We aim to combine (7) and (9). This yields our main subject of study.

Definition Let $f \in \mathbb{Q}[x]$ be an integer-valued polynomial of degree $d \geq 1$ such that $f(y) \geq 2$ and $f'(y) \geq 1$ for all $y \geq 1$. Then

$$V_f(x, y) := \sum_{k \leq y} f'(k) \sum_{l=1}^{f(k)} E(x; f(k), l)^2. \tag{12}$$

We will fix the polynomial f of degree $d \geq 1$ for the remainder of the text, alongside positive x and y which we assume to be sufficiently large and to satisfy the condition $f(y) \leq x$. Our aspiration is to prove the following asymptotic:

Theorem 2.1 *Let $x > 0$ and $y > 0$ such that $f(y) < x$. Then*

$$V_f(x, y) = C_f x^{1/2} f(y)^{3/2} + O\left(x^{1/4} f(y)^{7/4}\right) + O\left(xy^{d-1}\right) + O\left(x^{3/2+\varepsilon}\right),$$

where C_f is a certain constant depending on f only.

Note that the main term is dominant for

$$\max\left\{x^{2/3+\varepsilon}, x^{d/(d+2)}\right\} \ll f(y) \leq x.$$

That is, for $1 \leq \deg f \leq 4$ the main term is dominant in the range

$$x^{2/3+\varepsilon} \ll f(y) \leq x,$$

and for $\deg f \geq 5$, it is dominant for

$$x^{d/(d+2)} \ll f(y) \leq x.$$

Most of the proof will be achieved by combining the methods of J. Brüdern and T. D. Wooley [BW11] with those of R. C. Vaughan [Vau05], whose work itself relies heavily on earlier papers [BGP+98, Vau98a, Vau98b]. Note that for the sake of convenience, we restricted our investigations to squarefree numbers, although the methods presented could just as well be adapted for k-free numbers.

We will establish the asymptotic in Thm. 2.1 through a series of propositions. But first, let us take care of small values for y. From the error bound (4) we see immediately that

$$V_f(x,y) \ll x \sum_{k\le y} f(k)f'(k) \ll xf(y)^2.$$

Now let y_1 denote the unique $y>0$ such that $f(y)=x^{1/4}$. Hence

$$V_f(x,y) = V_f'(x,y) + V_f(x,y_1) = V_f'(x,y) + O\left(x^{3/2}\right),$$

where we wrote

$$V_f'(x,y) := \sum_{y_1<k\le y} f'(k) \sum_{l=1}^{f(k)} E(x;f(k),l)^2$$

for the truncated sum. The next step is then to isolate those sums that contribute the main terms to $V_f(x,y)$.

Proposition 2.2 *We have*

$$V_f(x,y) = \zeta(2)^{-1}xf(y) + 2S_0(x,y) - x^2\Phi_f(y) + O\left(x^{3/2+\varepsilon}\right) + O\left(xy^{d-1}\right),$$

where

$$S_0(x,y) = \sum_{y_1<k\le y} f'(k) \sum_{n\le x} \sum_{\substack{m<n \\ m\equiv n \ (f(k))}} \mu(n)^2\mu(m)^2, \tag{13}$$

and

$$\Phi_f(y) = \sum_{y_1<k\le y} f'(k) \sum_{r|f(k)} g(f(k),r)^2\varphi(f(k)/r). \tag{14}$$

Proving this requires only straightforward rearrangements which are presented in Chapter 3. The asymptotic behaviour of $V_f(x,y)$ is thus determined by $S_0(x,y)$ and $\Phi_f(y)$, and is subject to cancellations in the terms of these sums. As we will see, this largely depends on some arithmetic functions we want to define now.

Definition Let $G(n)$ denote the multiplicative function defined by

$$G(p^t) := \begin{cases} -(p^2-1)^{-1}, & t=1,2, \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

for prime powers p^t. For integers q and h we define

$$w_h(q) := \frac{1}{q}\sum_{a=1}^{q} c_q(hf(a)), \tag{16}$$

where $c_q(n)$ denotes Ramanujan's sum. We sum up these two functions in the series

$$W(h) := \sum_{q=1}^{\infty} G(q)^2 w_h(q). \tag{17}$$

With these means, we will then reduce $S_0(x,y)$ to another sum.

Proposition 2.3 *We have*

$$2S_0(x,y) = \zeta(2)^{-2}\left(f(y_1)^2\Theta_f(x/f(y_1)) - f(y)^2\Theta_f(x/f(y))\right) + O\left(x^{3/2+\varepsilon}\right) + O\left(xy^{d-1}\right),$$

where

$$\Theta_f(H) = \sum_{h\le H} \frac{W(h)}{h}(H-h)^2. \tag{18}$$

These deductions involve the circle method and will be discussed in Chapter 4. The sum $\Theta_f(H)$ will be analysed by the properties of the following analytic function:

Definition Let $\Re s > 0$. We then define

$$D(s) = \sum_{n=1}^{\infty} W(n) n^{-s-1}. \tag{19}$$

We will see that this converges to an analytic function in the domain. Indeed we further have:

Proposition 2.4 *The function $D(s)$ has a meromorphic continuation to the half-plane $\Re s > -2$, where the only poles in $\Re s \ge -\frac{7}{4}$ are single poles at $s = 0$ and $s = -\frac{3}{2}$.*

A simple application of Perron's formula and the residue theorem then gives us the asymptotic for $S_0(x,y)$.

Corollary 2.5 *We have*

$$\begin{aligned} 2S_0(x,y) = \zeta(2)^{-2}\Gamma_0 x^2 \log(f(y)/f(y_1)) - \zeta(2)^{-1} x f(y) + C_f x^{1/2} f(y)^{3/2} \\ + O\left(x^{1/4} f(y)^{7/4}\right) + O\left(x^{3/2+\varepsilon}\right) + O\left(xy^{d-1}\right), \end{aligned}$$

where Γ_0 is a certain constant depending on f only.

These examinations are subject of Chapter 6. All that remains is then the analysis of $\Phi_f(y)$, which will cancel the term containing Γ_0.

Proposition 2.6 *We have*

$$\Phi_f(y) = \zeta(2)^{-2}\Gamma_0 \log\left(f(y)/f(y_1)\right) + O\left(y^{-2d+1}\right).$$

The proof of this will be presented in Chapter 7. Combining Prop. 2.2 with Cor. 2.5 and Prop. 2.6 thus completes the proof of Thm. 2.1.

3 Analysing V_f by Elementary Rearrangements

The first step of the proof simply consists of rearranging the sums in $V_f(x,y)$, and isolating those that contribute the main term. This requires only elementary methods that are presented in this chapter.

So let us write out the definition (3) of $E(x;k,l)$ and open the square in the definition (12) of $V_f(x,y)$. We obtain

$$V_f'(x,y) = S_1(x,y) - 2xS_2(x,y) + x^2 S_3(y), \tag{20}$$

where

$$S_1(x,y) = \sum_{y_1<k\le y} f'(k) \sum_{l=1}^{f(k)} \sum_{\substack{n\le x \\ n\equiv l \ (f(k))}} \sum_{\substack{m\le x \\ m\equiv l \ (f(k))}} \mu(n)^2\mu(m)^2,$$

$$S_2(x,y) = \sum_{y_1<k\le y} f'(k) \sum_{l=1}^{f(k)} g(f(k),l)Q(x;f(k),l),$$

and

$$S_3(y) = \sum_{y_1<k\le y} f'(k) \sum_{l=1}^{f(k)} g(f(k),l)^2.$$

First notice in $S_1(x,y)$ that by summing over all residue classes, we eventually sum over those values of n and m that are congruent modulo $f(k)$. I. e.,

$$S_1(x,y) = \sum_{y_1<k\le y} f'(k) \sum_{n\le x} \sum_{\substack{m\le x \\ m\equiv n \ (f(k))}} \mu(n)^2\mu(m)^2.$$

Separating diagonal from off-diagonal terms hence yields

$$S_1(x,y) = \sum_{y_1<k\le y} f'(k) \sum_{n\le x} \mu(n)^2 + 2 \sum_{y_1<k\le y} f'(k) \sum_{n\le x} \sum_{\substack{m< n \\ m\equiv n \ (f(k))}} \mu(n)^2\mu(m)^2.$$

We apply Euler's summation formula (cf. [Brü95, (4.5)]) to the first sum, and obtain

$$\sum_{y_1<k\le y} f'(k) = \int_{y_1}^{y} f'(t)\,\mathrm{d}t + O\left(\int_{y_1}^{y} |f''(t)|\,\mathrm{d}t + f'(y_1) + f'(y)\right) = f(y) + O\left(y^{d-1}\right).$$

Using the asymptotic (3) for $Q(x)$ yields

$$\sum_{n\leq x}\mu(n)^2 = Q(x) = \zeta(2)^{-1}x + O\left(x^{1/2}\right).$$

On remembering the definition (13) of $S_0(x,y)$ we arrive at

$$S_1(x,y) = \zeta(2)^{-1}xf(y) + 2S_0(x,y) + O\left(xy^{d-1}\right) + O\left(x^{1/2}y^d\right). \tag{21}$$

Before we proceed to analyse $S_2(x,y)$, we need some properties on the asymptotic density $g(k,l)$.

Lemma 3.1 *The asymptotic density of $Q(x;k,l)$ depends only on the greatest common divisor of k and l rather than on l, i. e.,*

$$g(k,l) = g(k,(k,l)).$$

Proof. This follows directly from (11) by the observation that those $m \in \mathbb{N}$ with $(m^2,k) \mid l$ are precisely those with $(m^2,k) \mid (k,l)$. □

Lemma 3.2 ([Vau98b, Lemma 2.2]) *Let $k \in \mathbb{N}$ and $r \mid k$. Then*

$$\sum_{t\mid k/r}\mu(t)g(rt,rt) = g(k,r)\varphi(k/r).$$

Proof. We first show that

$$g(r,r) = \sum_{\substack{a=1\\ r\mid a}}^{q} g(q,(q,a)) \tag{22}$$

for $q \geq r$. Sorting according to the residue classes of n modulo q yields

$$Q(x;r,r) = \sum_{\substack{n\leq x\\ r\mid n}}\mu(n)^2 = \sum_{\substack{a=1\\ r\mid a}}^{q}\sum_{\substack{n\leq x\\ n\equiv a\ (q)}}\mu(n)^2 = \sum_{\substack{a=1\\ r\mid a}}^{q} Q(x;q,a).$$

Now with (2) and Lemma 3.1, we obtain

$$g(r,r) = \lim_{x\to\infty}\frac{Q(x;r,r)}{x} = \sum_{\substack{a=1\\ r\mid a}}^{q}\lim_{x\to\infty}\frac{Q(x;q,a)}{x} = \sum_{\substack{a=1\\ r\mid a}}^{q} g(q,a) = \sum_{\substack{a=1\\ r\mid a}}^{q} g(q,(q,a))$$

as claimed in (22). Using this identity with rt instead of r we find that

$$\begin{aligned}\sum_{t|k/r}\mu(t)g(rt,rt) &= \sum_{t|k/r}\mu(t)\sum_{\substack{a=1\\ rt|a}}^{k} g(k,(k,a))\\ &= \sum_{\substack{b=1\\ (b,k/r)=1}}^{k/r} g(k,r)\\ &= g(k,r)\varphi(k/r).\end{aligned}$$

□

Lemma 3.3 ([Vau05, Lemma 2.4]) *Let $k \in \mathbb{N}$. Then*

$$\sum_{l=1}^{k} g(k,l)^2 = \sum_{r|k} g(k,r)^2\varphi(k/r).$$

Proof. This follows immediately by using Lemma 3.1:

$$\sum_{l=1}^{k} g(k,l)^2 = \sum_{r|k}\sum_{\substack{l=1\\ (l,k)=r}}^{k} g(k,(k,l))^2 = \sum_{r|k} g(k,r)^2 \sum_{\substack{l=1\\ (l,k/r)=1}}^{k/r} 1 = \sum_{r|k} g(k,r)^2\varphi(k/r).$$

□

Applying now Lemma 3.1 to $S_2(x,y)$ and sorting according to the values of the gcd, we obtain

$$\begin{aligned}S_2(x,y) &= \sum_{y_1<k\le y} f'(k)\sum_{l=1}^{f(k)} g(f(k),(f(k),l))Q(x;f(k),l)\\ &= \sum_{y_1<k\le y} f'(k)\sum_{r|f(k)} g(f(k),r)\sum_{\substack{l=1\\ (l,f(k)/r)=1}}^{f(k)/r} Q(x;f(k),lr).\end{aligned}$$

Plugging in the definition (1) of $Q(x;k,l)$ yields

$$\begin{aligned}S_2(x,y) &= \sum_{y_1<k\le y} f'(k)\sum_{r|f(k)} g(f(k),r)\sum_{\substack{l=1\\ (l,f(k)/r)=1}}^{f(k)/r}\ \sum_{\substack{n\le x\\ n\equiv l\ (f(k)/r)}} \mu(n)^2\\ &= \sum_{y_1<k\le y} f'(k)\sum_{r|f(k)} g(f(k),r)\sum_{\substack{n\le x\\ (n,f(k)/r)=1}} \mu(nr)^2\\ &= \sum_{y_1<k\le y} f'(k)\sum_{r|f(k)} g(f(k),r)\sum_{t|f(k)/r}\mu(t)\sum_{n\le x/rt}\mu(nrt)^2.\end{aligned}$$

We can now apply Lemma 1.1 to the innermost sum. Thus

$$\sum_{n\leq x/rt} \mu(nrt)^2 = Q(x;rt,rt) = g(rt,rt)x + O\left(\sigma_0(rt)(x/rt)^{1/2}\right).$$

The error term then contributes to the sum

$$\ll \sum_{y_1<k\leq y} f'(k) \sum_{r|f(k)} g(f(k),r) \sum_{t|f(k)/r} \sigma_0(rt)(x/rt)^{1/2}.$$

We have the obvious bound

$$Q(x;k,l) \leq \sum_{\substack{n\leq x\\ n\equiv l\ (k)}} 1 \leq \frac{x}{k} + 1,$$

and hence by (2),

$$g(k,l) \leq \lim_{x\to\infty} \frac{1+x/k}{x} = \frac{1}{k}.$$

This yields an error bounded by

$$\ll x^{1/2} \sum_{y_1<k\leq y} \frac{f'(k)}{f(k)} \sum_{l|f(k)} \sigma_0(l)^2 l^{-1/2} \ll x^{1/2}\log y \ll x^{1/2+\varepsilon}.$$

Applying Lemma 3.2, we obtain

$$S_2(x,y) = x \sum_{y_1<k\leq y} f'(k) \sum_{r|f(k)} g(f(k),r)^2 \varphi(f(k)/r) + O\left(x^{1/2+\varepsilon}\right).$$

Remembering the definition (14) of $\Phi_f(y)$, this finally yields

$$S_2(x,y) = x\Phi_f(y) + O\left(x^{1/2+\varepsilon}\right). \tag{23}$$

For the analysis of $S_3(y)$ we simply apply Lemma 3.3, and obtain

$$S_3(y) = \sum_{y_1<k\leq y} f'(k) \sum_{r|f(k)} g(f(k),r)^2 \varphi(f(k)/r) = \Phi_f(y). \tag{24}$$

Now plugging the formulae (21), (23), and (24) for $S_1(x,y)$, $S_2(x,y)$, and $S_3(y)$, respectively, into the expression (20) for $V_f'(x,y)$ yields

$$\begin{aligned} V_f(x,y) &= V_f'(x,y) + O\left(x^{3/2}\right) \\ &= \zeta(2)^{-1} x f(y) + 2S_0(x,y) - x^2\Phi_f(y) + O\left(x^{3/2+\varepsilon}\right) + O\left(xy^{d-1}\right), \end{aligned}$$

concluding the proof of Prop. 2.2. We can thus proceed with the analysis of $S_0(x,y)$.

4 Analysing S_0 by the Circle Method

The sum $S_0(x,y)$ is a classic object accessible by the Hardy–Littlewood circle method. This first transforms the sum into an integral over the unit circle (usually parameterised by the unit interval), and then partitions it into the major arcs (values close to certain rationals) and the minor arcs. The hope is then that the major arcs, despite having small content, contribute the main term to the sum, whereas the minor arcs can be bound by a sufficiently small error term.

In order to apply the circle method to $S_0(x,y)$, we need two functions that help transforming the sum into an integral.

Definition Let $\alpha \in \mathbb{R}$. Then

$$T_f(\alpha) := \sum_{y_1<k\le y} f'(k) \sum_{h\le x/f(k)} e(\alpha h f(k)), \tag{25}$$

and

$$U(\alpha) := \sum_{n\le x} \mu(n)^2 e(\alpha n), \tag{26}$$

where we wrote for short

$$e(\alpha) := \exp(2\pi i\alpha).$$

Note that these functions as the linear combination of 1-periodic functions are 1-periodic themselves.

By orthogonality it is then obvious that

$$S_0(x,y) = \int_0^1 T_f(\alpha)|U(\alpha)|^2 \,\mathrm{d}\alpha.$$

For the Farey dissection we need a parameter $R > 0$ to be chosen "small" in comparison with x. It turns out that $R = \frac{1}{2}x^{1/2}$ is a suitable choice for our purpose. In what follows, we shall thus assume this value for R.

Definition For $T > 0$ let

$$\mathfrak{M}_T(q,a) := \left\{ \alpha \in \mathbb{R} : |q\alpha - a| \le \frac{T}{x} \right\}. \tag{27}$$

The *major arcs* $\mathfrak{M}_T$ are the pointwise disjoint union of the $\mathfrak{M}_T(q,a)$ with $1 \le a \le q \le T$ and $(a,q) = 1$. Accordingly, we define the *minor arcs* to be

$$\mathfrak{m}_T := (T/x, 1 + T/x] \setminus \mathfrak{M}_T.$$

We will see that the value of $T_f(\alpha)$ on the minor arcs $\mathfrak{M}_R$ is not too large. More precisely we have:

Lemma 4.1 *Let $R = \frac{1}{2}x^{1/2}$. Then*

$$\sup_{\alpha\in\mathfrak{m}_R} |T_f(\alpha)| \ll \frac{x}{R}\log x \ll x^{1/2+\varepsilon}.$$

We will return to prove this in Chapter 5. Using the obvious bound

$$\int_0^1 |U(\alpha)|^2 \,\mathrm{d}\alpha = \sum_{n\le x} \mu(n)^2 \ll x$$

we can conclude

$$S_0(x,y) = S_{\mathfrak{M}_R}(x,y) + O\left(x^{3/2+\varepsilon}\right), \tag{28}$$

where we wrote

$$S_{\mathfrak{M}_R}(x,y) = \int_{\mathfrak{M}_R} T_f(\alpha)|U(\alpha)|^2 \,\mathrm{d}\alpha = \sum_{q\le R} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{-R/qx}^{R/qx} T_f(\beta+a/q)|U(\beta+a/q)|^2 \,\mathrm{d}\beta. \tag{29}$$

The next step is to reduce $U(\alpha)$ to a simpler function.

Definition Let $\alpha \in \mathbb{R}$. Then

$$J(\alpha) := \sum_{n\le x} e(n\alpha). \tag{30}$$

The following arithmetic function will play an important role as it encodes the behaviour of $g(k,l)$ with an exponential sum:

$$\nu(q) := \sum_{a=1}^{q} g(q,a)e(a/q). \tag{31}$$

Henceforth, we want to work with

$$U^*(\alpha;q,a) := \nu(q)J(\alpha - a/q) \tag{32}$$

rather than $U(\alpha)$. It turns out that the difference

$$\Delta(\alpha;q,a) := \begin{cases} U(\alpha) - U^*(\alpha;q,a), & \alpha \in \mathfrak{M}_R(q,a), \\ 0, & \alpha \in \mathfrak{m}_R, \end{cases} \tag{33}$$

is rather small, and can thus be neglected asymptotically. Note that the arguments q and a in $U^*(\alpha;q,a)$ and $\Delta(\alpha;q,a)$ are actually implied by $\alpha \in \mathfrak{M}_R(q,a)$, and are occasionally dropped for that reason.

Rearranging the definition (33) of $\Delta(\alpha; q, a)$ we obtain

$$\begin{aligned}|U(\alpha)|^2 &= |U^*(\alpha)|^2 + 2\Re(U^*(\alpha)\overline{\Delta}(\alpha)) + |\Delta(\alpha)|^2 \\ &= |U^*(\alpha)|^2 + O\left(|U^*(\alpha)\Delta(\alpha)|\right) + O\left(|\Delta(\alpha)|^2\right),\end{aligned}$$

where $\overline{z}$ denotes complex conjugation. We now want to present the rigorous bound of $\Delta(\alpha; q, a)$ on the major arcs.

Lemma 4.2 ([BGP+98, Lemma 3.2]) *Let T be a real number with $1 \leq T \leq \frac{1}{2}x^{1/2}$. Then*

$$\int_{\mathfrak{M}_T} |\Delta(\alpha)|^2 \,\mathrm{d}\alpha = \sum_{q\leq T} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}_T} |\Delta(\alpha; q, a)|^2 \,\mathrm{d}\alpha \ll x^{\varepsilon} T^2.$$

A similar estimate holds for the cross-term.

Lemma 4.3 ([BGP+98, Lemma 4.2]) *Let T be a real number with $1 \leq T \leq \frac{1}{2}x^{1/2}$. Then*

$$\int_{\mathfrak{M}_T} |U^*(\alpha)\Delta(\alpha)| \,\mathrm{d}\alpha = \sum_{q\leq T} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}_T} |U^*(\alpha; q, a)\Delta(\alpha; q, a)| \,\mathrm{d}\alpha \ll x^{1/2+\varepsilon} T^{3/4}.$$

We can now infer bounds for the integrals that do not contribute to the main term.

Corollary 4.4 *With $R = \frac{1}{2}x^{1/2}$ we have the bounds*

$$\int_{\mathfrak{M}_R} T_f(\alpha)|\Delta(\alpha)|^2 \,\mathrm{d}\alpha \ll x^{3/2+\varepsilon}$$

and

$$\int_{\mathfrak{M}_R} T_f(\alpha)|U^*(\alpha)\Delta(\alpha)| \,\mathrm{d}\alpha \ll x^{3/2+\varepsilon}.$$

We will present these more technical proofs in Chapter 5. Applying then Corollary 4.4 to the definition (29) of $S_{\mathfrak{M}_R}(x, y)$ yields

$$S_{\mathfrak{M}_R}(x, y) = \sum_{q\leq R} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{-R/qx}^{R/qx} T_f(\beta + a/q)|U^*(\beta + a/q)|^2 \,\mathrm{d}\beta + O\left(x^{3/2+\varepsilon}\right).$$

We thus return to the analysis of $U^*(\alpha; q, a)$. First, we will simplify the function $\nu(q)$.

Lemma 4.5 ([Vau05, Lemma 2.5]) *Let $q \in \mathbb{N}$. Then*

$$\nu(q) = \zeta(2)^{-1} G(q),$$

where $G(q)$ is the multiplicative function defined in (15).

Proof. By the equation (11) for $g(k,l)$ and the definition (31) of $\nu(q)$, we find that

$$\begin{aligned}\nu(q) &= \sum_{r|q} \sum_{\substack{a=1\\(a,q)=r}}^{q} g(q,r) e(a/q)\\ &= \sum_{r|q} g(q,r)\mu(q/r)\\ &= \sum_{r|q} \mu(q/r) \sum_{\substack{m=1\\(m^2,q)|r}}^{\infty} \frac{\mu(m)(m^2,q)}{m^2 q}.\end{aligned}$$

Sorting according to values of the gcd yields

$$\nu(q) = \sum_{\substack{m=1\\q|m^2}}^{\infty} \frac{\mu(m)}{m^2}.$$

The sum is obviously zero if q is not cubefree, confirming the claimed identity. Assuming now that q is cubefree we find the Euler product of the above as

$$\nu(q) = \prod_{p \nmid q}(1-p^{-2}) = \prod_{p}(1-p^{-2}) \prod_{p|q}(1-p^{-2})^{-1} = \zeta(2)^{-1} G(q). \qquad \square$$

Hence the main term of $S_{\mathfrak{M}_R}(x,y)$ takes the shape

$$\zeta(2)^{-2} \sum_{q \le R} G(q)^2 \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{-R/qx}^{R/qx} T_f(\beta + a/q) |J(\beta)|^2 \,\mathrm{d}\beta. \tag{34}$$

We would now like to extend the range of integration to $|\beta| \le \frac{1}{2}$. Note that in the range $R/qx \le |\beta| \le \frac{1}{2}$ we have $J(\beta) \ll |\beta|^{-1}$. Using

$$T_f(\beta) \ll x \sum_{y_1 < k \le y} \frac{f'(k)}{f(k)} \ll x \log x,$$

we find that

$$\begin{aligned}
&\zeta(2)^{-2}\sum_{q\le R} G(q)^2 \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{R/qx\le|\beta|\le\frac{1}{2}} T_f(\beta+a/q)|J(\beta)|^2\,\mathrm{d}\beta\\
&\ll x(\log x)\sum_{q\le x} G(q)^2 \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{R/qx}^{1/2} \beta^{-2}\,\mathrm{d}\beta\\
&\ll \frac{x^2}{R}(\log x)\sum_{q\le x} G(q)^2\varphi(q) \ll \frac{x^2}{R}\log x \ll x^{3/2+\varepsilon}.
\end{aligned}$$

So we can indeed extend the range of integration with acceptable errors. Using orthogonality we thus obtain for the integral

$$\begin{aligned}
\int_{-1/2}^{1/2} T_f(\beta+a/q)|J(\beta)|^2\,\mathrm{d}\beta &= \sum_{y_1<k\le y} f'(k) \sum_{h\le x/f(k)} \sum_{n\le x} \sum_{\substack{m\le x\\ n-m=hf(k)}} e\left(\frac{ahf(k)}{q}\right)\\
&= \sum_{y_1<k\le y} f'(k) \sum_{h\le x/f(k)} e\left(\frac{ahf(k)}{q}\right)(\lfloor x\rfloor - hf(k)).
\end{aligned}$$

Note that $\lfloor x\rfloor = x + O(1)$, so we may replace $\lfloor x\rfloor$ by x, introducing an error bounded by

$$\ll x \sum_{y_1<k\le y} \frac{f'(k)}{f(k)} \ll x\log x.$$

Plugging these results into the expression (34) for $S_{\mathfrak{M}_R}(x,y)$ yields

$$S_{\mathfrak{M}_R}(x,y) = \sum_{q\le R}\frac{G(q)^2}{\zeta(2)^2} \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{y_1<k\le y} f'(k) \sum_{h\le x/f(k)} e\left(\frac{ahf(k)}{q}\right)(x-hf(k)) + O\left(x^{3/2+\varepsilon}\right).$$

Exchanging the order of summation and writing

$$c_q(n) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{an}{q}\right)$$

for Ramanujan's sum, we obtain

$$S_{\mathfrak{M}_R}(x,y) = \zeta(2)^{-2}M_0(x,y) + O\left(x^{3/2+\varepsilon}\right), \tag{35}$$

where

$$M_0(x,y) = \sum_{q\le R} G(q)^2 \sum_{y_1<k\le y} f'(k) \sum_{h\le x/f(k)} c_q(hf(k))(x-hf(k)). \tag{36}$$

Define $y(h)$ by means of the equation $f(y(h)) = \min\{f(y), x/h\}$. This helps to exchange the order of summation of h and k:

$$M_0(x,y) = \sum_{q\le R} G(q)^2 \sum_{h\le x/f(y_1)} \sum_{y_1<k\le y(h)} f'(k)c_q(hf(k))(x-hf(k)).$$

Since $c_q(hf(k))$ depends only on the residue class of $f(k)$ modulo q and thus on the residue class of k modulo q, we sort the summation according to these residue classes.

$$M_0(x,y) = \sum_{q\le R} G(q)^2 \sum_{h\le x/f(y_1)} \sum_{a=1}^{q} c_q(hf(a)) \sum_{\substack{y_1<k\le y(h)\\ k\equiv a\ (q)}} f'(k)(x-hf(k)). \tag{37}$$

We can now apply Euler's summation formula to the innermost sum.

Lemma 4.6 ([BW11, Lemma 4.1]) *Let f, x, y, and h as above. Then*

$$\sum_{\substack{y_1<k\le y(h)\\ k\equiv a\ (q)}} f'(k)(x-hf(k)) = \frac{1}{q}\int_{f(y_1)}^{f(y(h))} x-ht\,\mathrm{d}t + O\left(xy^{d-1}\right) + O\left(y^{2d-1}h\right).$$

Proof. Let $F:[X,Y]\to\mathbb{R}$ be a smooth function. For integers a and q with $q\neq 0$ we have by Euler's summation formula

$$\sum_{\substack{X<k\le Y\\ k\equiv a\ (q)}} F(k) = \frac{1}{q}\int_X^Y F(t)\,\mathrm{d}t + E, \tag{38}$$

where the error term satisfies the bound

$$|E| \le \int_X^Y |F'(t)|\,\mathrm{d}t + |F(X)| + |F(Y)|. \tag{39}$$

So we now choose

$$F(t) = f'(t)(x-ht) \ll xt^{d-1} + ht^{2d-1} \tag{40}$$

for the function, together with $X = y_1$ and $Y = y(h)$ for the range. By substitution, we obtain immediately

$$\int_{y_1}^{y(h)} f'(t)(x-hf(t))\,\mathrm{d}t = \int_{f(y_1)}^{f(y(h))} x-ht\,\mathrm{d}t,$$

confirming the main term.

In order to examine the error term, we need the derivative

$$F'(t) = f''(t)(x-hf(t)) - hf'(t)^2 \ll xt^{d-2} + ht^{2d-2}. \tag{41}$$

Using the bounds (40) for $F(t)$ and (41) for $F'(t)$ in equation (39) for the error term, we arrive at

$$E \ll xy^{d-1} + hy^{2d-1}.$$

Combining this with Euler's summation formula (38) concludes the proof. □

Plugging Lemma 4.6 into (37) yields an error bounded by

$$\ll \sum_{q \le R} G(q)^2 \sum_{h \le x/f(y_1)} q\varphi(q)(xy^{d-1} + y^{2d-1}h) \ll xy^{d-1}.$$

On remembering the definition (16) of $w_h(q)$ and exchanging the order of summation we find that

$$\begin{aligned} M_0(x,y) &= \sum_{q \le R} G(q)^2 \sum_{h \le x/f(y_1)} w_h(q) \int_{f(y_1)}^{f(y(h))} x - ht \,\mathrm{d}t + O\left(xy^{d-1}\right) \\ &= \sum_{h \le x/f(y_1)} \int_{f(y_1)}^{f(y(h))} x - ht \,\mathrm{d}t \sum_{q \le R} G(q)^2 w_h(q) + O\left(xy^{d-1}\right). \end{aligned}$$

We now want to extend the summation over q to all natural numbers. For this, we first need some properties of the function $w_h(q)$.

Lemma 4.7 ([BW11, Lemma 4.2]) *Let $h \in \mathbb{N}$ be fixed. Then the function $w_h(q)$ as defined in (16) is multiplicative of q. If further p is a prime, we have*

$$w_h(p) = \begin{cases} p-1, & p \mid h, \\ \varrho(p) - 1, & p \nmid h, \end{cases}$$

and

$$w_h(p^2) = \begin{cases} p^2 - p, & p^2 \mid h, \\ p\varrho(p) - p, & p \parallel h, \\ \varrho(p^2) - \varrho(p), & p \nmid h, \end{cases}$$

where $\varrho(m)$ denotes the number of solutions of the congruence $f(a) \equiv 0 \ (m)$ with $0 \le a < m$.

Proof. It is a well-known fact that $c_q(n)$ is a multiplicative function of q (cf. [HW79, Thm. 272]). Moreover, the value of $c_q(n)$ only depends on the residue class of n modulo q. So assume that q and q' are coprime. Then we have

$$w_h(qq') = \frac{1}{qq'} \sum_{a=1}^{qq'} c_{qq'}(hf(a)) = \frac{1}{qq'} \sum_{a=1}^{qq'} c_q(hf(a)) c_{q'}(hf(a)).$$

Now we can apply the Chinese Remainder Theorem to the residue class of $f(a)$ modulo qq', and obtain

$$w_h(qq') = \left(\frac{1}{q}\sum_{a=1}^{q} c_q(hf(a))\right)\cdot\left(\frac{1}{q'}\sum_{a=1}^{q'} c_{q'}(hf(a))\right) = w_h(q)w_h(q').$$

Hence we need to calculate the values at prime powers p^t. Writing out the definitions and completing Ramanujan's sum yields

$$w_h(p^t) = \frac{1}{p^t}\sum_{a=1}^{p^t}\sum_{\substack{b=1\\(b,p^t)=1}}^{p^t} e\left(\frac{bhf(a)}{p^t}\right) = \frac{1}{p^t}\sum_{a=1}^{p^t}\left(\sum_{b=1}^{p^t} e\left(\frac{bhf(a)}{p^t}\right) - \sum_{b=1}^{p^{t-1}} e\left(\frac{bhf(a)}{p^{t-1}}\right)\right).$$

Recall the basic exponential sum property

$$\sum_{b=1}^{n} e\left(\frac{bq}{n}\right) = \begin{cases} n, & n \mid q,\\ 0, & n \nmid q,\end{cases}$$

which follows directly from the geometric series. As the function $G(p^t)$ is zero for $t > 2$ we are only interested in the values at p and p^2. So using the above property we obtain

$$w_h(p) = \left(\sum_{\substack{a=1\\ hf(a)\equiv 0\ (p)}}^{p} 1\right) - 1 = \begin{cases} p-1, & p \mid h,\\ \varrho(p)-1, & p\nmid h.\end{cases}$$

The same argument can be used to deduce that

$$w_h(p^2) = \left(\sum_{\substack{a=1\\ hf(a)\equiv 0\ (p^2)}}^{p^2} 1\right) - \left(\sum_{\substack{a=1\\ hf(a)\equiv 0\ (p)}}^{p} 1\right) = \begin{cases} p^2-p, & p^2 \mid h,\\ p\varrho(p)-p, & p \parallel h,\\ \varrho(p^2)-\varrho(p), & p\nmid h.\end{cases} \qquad \square$$

Note that $\varrho(m)$ is multiplicative by the Chinese Remainder Theorem and that the number of solutions $\varrho(p^l)$ for prime powers is bounded by $\varrho(p^l) \le d^l$. Thus we infer that the multiplicative function $w_h(q)$ is bounded by $O\left((q,h)q^{\varepsilon}\right)$ for cubefree q. In the non-cubefree case the value of $G(q)$ will vanish, establishing the same bound. Hence

$$\sum_{q\le R} G(q)^2 w_h(q) = W(h) + O\left(h^{\varepsilon}R^{-3}\right) = W(h) + O\left(x^{-3/2}h^{\varepsilon}\right),$$

where $W(h)$ is the series defined in (17). Obviously we have

$$\int_{f(y_1)}^{f(y(h))} x - ht\,\mathrm{d}t \ll f(y)x \ll x^2.$$

This yields

$$M_0(x,y) = \sum_{h \le x/f(y_1)} W(h) \int_{f(y_1)}^{f(y(h))} x - ht \,\mathrm{d}t + O\left(x^{3/2+\varepsilon}\right) + O\left(xy^{d-1}\right).$$

We now evaluate the integral by noting that the antiderivative of $x - ht$ with respect to t is $t(x - \frac{1}{2}ht)$. Remember that $f(y(h)) = f(y)$ for $h \le x/f(y)$ and $f(y(h)) = x/h$ for $h > x/f(y)$. Splitting the sum at $h = x/f(y)$ we find that

$$\begin{aligned} M_0(x,y) = \frac{1}{2} \sum_{h \le x/f(y)} &\frac{W(h)}{h} \left(2xf(y)h - f(y)^2h^2 - 2xh + h^2 - (x-h)^2\right) \\ &+ \frac{1}{2} f(y_1)^2 \sum_{h \le x/f(y_1)} \frac{W(h)}{h} (x/f(y_1) - h)^2 + O\left(x^{3/2+\varepsilon}\right) + O\left(xy^{d-1}\right). \end{aligned}$$

Using $\Theta_f(H)$ as defined in (18) we obtain

$$2M_0(x,y) = f(y_1)^2\Theta_f(x/f(y_1)) - f(y)^2\Theta_f(x/f(y)) + O\left(x^{3/2+\varepsilon}\right) + O\left(xy^{d-1}\right).$$

In view of the equations (28) and (35) connecting $S_0(x,y)$, $S_{\mathfrak{M}_R}(x,y)$, and $M_0(x,y)$, we can justify the choice $R = \frac{1}{2}x^{1/2}$ as claimed to conclude the proof of Proposition 2.3. The next step is thus to analyse the asymptotic behaviour of $\Theta_f(H)$. But before we can turn our attention to that, we need to catch up on the pending proofs of this chapter.

5 Analysing the Minor and Major Arcs

This chapter is dedicated to the proofs of Lemmata 4.1 to 4.3 and Corollary 4.4 which we left out in Chapter 4. We will start with the proofs of Lemmata 4.2 and 4.3, but need some auxiliary results first.

Lemma 5.1 ([Gal70, Lemma 1]) *Let a_n be a (complex) sequence and*

$$A(\alpha) := \sum_{n\in\mathbb{Z}} a_n e(n\alpha)$$

be an absolutely convergent exponential series. Let $\vartheta > 0$. Then

$$\int_{-1/2\vartheta}^{1/2\vartheta} |A(\alpha)| \,\mathrm{d}\alpha \ll \int_{-\infty}^{\infty} \left| \vartheta^{-1} \sum_{t<n\le t+\vartheta} a_n \right|^2 \mathrm{d}t.$$

Proof. Define temporarily

$$B_\vartheta(t) = \vartheta^{-1} \sum_{|n-t|\le\frac{1}{2}\vartheta} a_n.$$

The integral on the right hand side then becomes

$$\int_{-\infty}^{\infty} |B_\vartheta(t)|^2 \,\mathrm{d}t.$$

Further put

$$F_\vartheta(t) = \begin{cases} \vartheta^{-1}, & |x| \le \frac{1}{2}\vartheta, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$B_\vartheta(t) = \sum_{n\in\mathbb{Z}} a_n F_\vartheta(t-n).$$

Taking Fourier transforms, we obtain

$$\widehat{B}_\vartheta(t) = A(t) \cdot \widehat{F}_\vartheta(t).$$

We assumed the series $A(\alpha)$ to be absolutely convergent, so $B_\vartheta(t)$ is a bounded integrable function, and hence square-integrable. We infer from Plancherel's Theorem,

$$\int_{-\infty}^{\infty} |B_\vartheta(t)|^2 \,\mathrm{d}t = \int_{-\infty}^{\infty} |A(t)\widehat{F}_\vartheta(t)|^2 \,\mathrm{d}t. \tag{42}$$

Note that

$$\widehat{F}_\vartheta(t) = \frac{\sin \pi\vartheta t}{\pi\vartheta t} \gg 1,$$

for $|t| \le \frac{1}{2\vartheta}$, so we can conclude the claimed bound from (42). □

Lemma 5.2 ([BGP+98, Lemma 3.1]) *Let $T > 0$. Then*

$$\sum_{T<q\le 2T} q|\nu(q)|^2 \ll T^{-1/2}(\log 2T)^2,$$

and

$$\sum_{q=1}^{\infty} \varphi(q)|\nu(q)|^2 = \zeta(2)^{-1}.$$

Proof. By Lemma 4.5, $\nu(q)$ is a multiplicative function whose values we can easily calculate by the means of $G(q)$. With the Euler product we can give the estimate

$$\sum_{q\le T} q^{3/2}|\nu(q)|^2 \le \prod_{p\le T}\left(1 + p^{3/2}\nu(p)^2 + p^3\nu(p^2)^2\right).$$

Using the definition (15) of $G(q)$, we arrive at

$$\sum_{T<q\le 2T} q|\nu(q)|^2 \ll T^{-1/2}\prod_{p\le 2T}\left(1+\frac{2}{p}\right) \ll T^{-1/2}(\log 2T)^2.$$

For the identity we again exploit Lemma 4.5 and the Euler product. We thus obtain

$$\begin{aligned}
\sum_{q=1}^{\infty} \varphi(q)|\nu(q)|^2 &= \zeta(2)^{-2}\sum_{q=1}^{\infty}\varphi(q)|G(q)|^2 \\
&= \zeta(2)^{-2}\prod_p\left(1+\frac{(p-1)+(p^2-p)}{(p^2-1)^2}\right) \\
&= \zeta(2)^{-2}\prod_p\left(\frac{p^2}{p^2-1}\right) = \zeta(2)^{-1}.
\end{aligned}$$

□

Lemma 5.3 *Let X, Y, α be real numbers such that $X, Y \ge 1$, and $|q\alpha - a| \le q^{-1}$ with $(a,q) = 1$. Then*

$$\sum_{h\le X} \min\left\{\frac{XY}{h}, \frac{1}{\|\alpha h\|}\right\} \ll \left(\frac{XY}{q} + X + q\right)\log(2Xq).$$

Proof. This is Lemma 2.2 of Vaughan [Vau97]. □

We are now prepared to tackle the bounds for the integral over $|\Delta(\alpha)|^2$ and $|U^*(\alpha)\Delta(\alpha)|$ on the major arcs.

Proof of Lemma 4.2. Define temporarily

$$L(\vartheta; q, a) := \int_{-1/2\vartheta}^{1/2\vartheta} |\Delta(\beta + q/a; q, a)|^2 \, \mathrm{d}\beta.$$

An inspection of the definition (33) of $\Delta(\alpha; q, a)$ together with the definitions (26) of $U(\alpha)$, (30) of $J(\alpha)$, and (32) of $U^*(\alpha; q, a)$ yields

$$\Delta(\beta + q/a; q, a) = \sum_{n=1}^{\infty} u(n; q, a) e(n\beta),$$

where we wrote

$$u(n; q, a) = \begin{cases} \mu(n)^2 e(an/q) - \nu(q), & 1 \leq n \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

Choosing $0 < \vartheta < x$ we can apply Lemma 5.1, and obtain

$$L(\vartheta; q, a) \ll \vartheta^{-2} \int_{-\infty}^{\infty} \left| \sum_{t < n \leq t + \vartheta} u(n; q, a) \right|^2 \mathrm{d}t. \tag{43}$$

Now we use (3) to obtain for positive z,

$$\sum_{n \leq z} \mu(n)^2 e(na/q) = \sum_{b=1}^{q} e(ab/q) \sum_{\substack{n \leq z \\ n \equiv b \ (q)}} \mu(n)^2 = \sum_{b=1}^{q} e(ab/q)(g(q, b)z + E(z; q, a)).$$

Assume now a and q to be coprime. Then both a and ab run through a complete coset representative modulo q. Moreover, by Lemma 3.1 the function $g(q, b)$ depends only on the gcd of q and b. Hence, with the definition (31) of $\nu(q)$,

$$\sum_{b=1}^{q} e(ab/q) g(q, b) = \sum_{b=1}^{q} e(b/q) g(q, (q, b)) = \nu(q). \tag{44}$$

Thus for $0 < z \leq x$,

$$\sum_{n \leq z} u(n; q, a) = \delta(z; q, a) + z\nu(q) - (z + O(1))\nu(q) = \delta(z; q, a) + O(\nu(q)),$$

where

$$\delta(z; q, a) := \sum_{b=1}^{q} e(ab/q) E(z; q, a).$$

Define now

$$z(t) = \begin{cases} t, & 0 \leq t \leq x, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$w(t) = \begin{cases} t+\vartheta, & -\vartheta < t \leq x-\vartheta, \\ x, & x-\vartheta < t \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{t<n\leq t+\vartheta} u(n;q,a) = \delta(w(t);q,a) - \delta(z(t);q,a) + O\left(\nu(q)\right)$$

for $-\vartheta < t \leq x$, whereas the sum is zero otherwise. Thus, with (43), we obtain

$$\begin{aligned} L(\vartheta;q,a) &\ll \vartheta^{-2} \int_{-\vartheta}^{x} |\nu(q)|^2 + |\delta(z(t);q,a)|^2 + |\delta(w(t);q,a)|^2 \,\mathrm{d}t \\ &\ll \vartheta^{-2} \left(x\,|\nu(q)|^2 + \vartheta\,|\delta(x;q,a)|^2 + \int_0^x |\delta(t;q,a)|^2 \,\mathrm{d}t \right). \end{aligned} \tag{45}$$

We now find

$$\begin{aligned} \sum_{q\leq T} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{-T/qx}^{T/qx} |\Delta(\beta+q/a;q,a)|^2 \,\mathrm{d}\beta &= \sum_{q\leq T} \sum_{\substack{a=1\\(a,q)=1}}^{q} L(qx/2T;q,a) \\ &\ll \max_{1\leq T'\leq T} \sum_{T'<q\leq 2T'} \sum_{\substack{a=1\\(a,q)=1}}^{q} L(qx/2T;q,a) \\ &\ll (\log 2T) \max_{1\leq T'\leq T} \Psi(x,T,T'), \end{aligned} \tag{46}$$

where

$$\Psi(x,T,T') := \sum_{T'<q\leq 2T'} \sum_{\substack{a=1\\(a,q)=1}}^{q} L(T'x/2T;q,a).$$

Using again the orthogonality of the additive characters we have

$$\sum_{a=1}^{q} |\delta(z;q,a)|^2 = q \sum_{b=1}^{q} |E(z;q,b)|^2.$$

We can now apply bound (45) for $L(\vartheta; q, a)$ and obtain with the definition (7) of $V(x,y)$,

$$\Psi(x,T,T') \ll \frac{T^2}{xT'^2} \sum_{T'<q\leq 2T'} q|\nu(q)|^2 + \frac{T}{x} V(x,2T') + \frac{T^2}{x^2 T'} \int_0^x V(t,2T')\,\mathrm{d}t.$$

Using the estimate (8) for $V(x,y)$ and applying Lemma 5.2, we find that

$$\Psi(x,T,T') \ll x^{-1+\varepsilon} T^2 T'^{-5/2} + x^{\varepsilon} T T' + x^{\varepsilon} T^2.$$

Hence

$$\max_{1\leq T'\leq T} \Psi(x,T,T') \ll x^{-1+\varepsilon} T^2 + x^{\varepsilon} T^2 \ll x^{\varepsilon} T^2.$$

Considering the bound (46), this concludes the proof. □

The corresponding proof for the integral over $|U^*(\alpha)\Delta(\alpha)|$ on the major arcs works in a similar fashion.

Proof of Lemma 4.3. Let $1 \leq T' \leq T$, and define $\mathfrak{M}_{T'}(q,a)$ and $\mathfrak{M}_{T'}$ accordingly to (27). Define further

$$\mathfrak{N}_{T'} := \mathfrak{M}_{2T'} \setminus \mathfrak{M}_{T'}. \tag{47}$$

We perform a dyadic dissection and apply the Cauchy-Schwarz inequality to obtain

$$\sum_{q\leq T} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}_T} |U^*(\alpha;q,a)\Delta(\alpha;q,a)|\,\mathrm{d}\alpha \ll (\log 2T) \max_{1\leq T'\leq T} \left(U_1(T')^{1/2} U_2(T')^{1/2} \right), \tag{48}$$

where

$$U_1(T') = \int_{\mathfrak{N}_{T'}} |U^*(\alpha;q,a)|^2\,\mathrm{d}\alpha,$$

and

$$U_2(T') = \int_{\mathfrak{M}_{2T'}} |\Delta(\alpha;q,a)|^2\,\mathrm{d}\alpha.$$

By the definitions (32) of $U^*(\alpha;q,a)$ and (47) of $\mathfrak{N}_{T'}$ we find that

$$U_1(T') \ll \sum_{T'<q\leq 2T'} \varphi(q)|\nu(q)|^2 \int_{-1/2}^{1/2} |J(\beta)|^2\,\mathrm{d}\beta + \sum_{1\leq q\leq T'} \varphi(q)|\nu(q)|^2 \int_{T'/qx}^{1/2} |J(\beta)|^2\,\mathrm{d}\beta.$$

We can now apply Lemma 5.2, and obtain

$$U_1(T') \ll xT'^{-1/2+\varepsilon}.$$

A direct application of Lemma 4.2 yields

$$U_2(T') \ll x^{\varepsilon} T'^2.$$

Plugging these bounds into (48) we arrive at

$$\sum_{q\le T} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}_T} |U^*(\alpha;q,a)\Delta(\alpha;q,a)| \,\mathrm{d}\alpha \ll (\log 2T) \max_{1\le T'\le T} \left(x^{1/2+\varepsilon} T'^{3/4+\varepsilon}\right)$$
$$\ll x^{1/2+\varepsilon} T^{3/4},$$

concluding the proof. □

Before we can proceed with appropriate bounds for $T_f(\alpha)$, we split the outer summation at $f(k) = x^{1/2}$, and sort according to whether the index is divisible by a positive integer q or not. Subsequently, we define the functions

$$T'_{f,q}(\alpha) := \begin{cases} \sum\limits_{\substack{y_1<k\le y\\ q|f(k)}} f'(k) \sum\limits_{h\le x/f(k)} e(\alpha h f(k)), & f(y)\le x^{1/2}, \\ \sum\limits_{\substack{k>y_1\\ f(k)\le x^{1/2}\\ q|f(k)}} f'(k) \sum\limits_{h\le x/f(k)} e(\alpha h f(k)) & \\ \qquad + \sum\limits_{\substack{h\le x^{1/2}\\ q|h}} \sum\limits_{\substack{k\le y(h)\\ f(k)>x^{1/2}}} f'(k) e(\alpha h f(k)), & f(y) > x^{1/2}, \end{cases} \tag{49}$$

and

$$T''_{f,q}(\alpha) := \begin{cases} \sum\limits_{\substack{y_1<k\le y\\ q\nmid f(k)}} f'(k) \sum\limits_{h\le x/f(k)} e(\alpha h f(k)), & f(y)\le x^{1/2}, \\ \sum\limits_{\substack{k>y_1\\ f(k)\le x^{1/2}\\ q\nmid f(k)}} f'(k) \sum\limits_{h\le x/f(k)} e(\alpha h f(k)) & \\ \qquad + \sum\limits_{\substack{h\le x^{1/2}\\ q\nmid h}} \sum\limits_{\substack{k\le y(h)\\ f(k)>x^{1/2}}} f'(k) e(\alpha h f(k)), & f(y) > x^{1/2}, \end{cases} \tag{50}$$

where we recall that $y(h)$ is defined by the condition $f(y(h)) = \min\{f(y), x/h\}$. Note that $T''_{f,q}(\alpha) = 0$ for $q = 1$. It is then obvious that

$$T_f(\alpha) = T'_{f,q}(\alpha) + T''_{f,q}(\alpha) \tag{51}$$

for any $q \in \mathbb{N}$. So it suffices to find the necessary upper bounds separately for $T'_{f,q}(\alpha)$ and $T''_{f,q}(\alpha)$ for an arbitrary q.

With these means we can find a bound for $T_f(\alpha)$ that will allow us a uniform bound on the minor arcs.

Lemma 5.4 ([Vau05, Lemma 2.10]) *Let* $(a,q)=1$ *and* $|q\alpha - a| \le q^{-1}$. *Then*

$$T_f(\alpha) \ll (xq^{-1}+q)\log x.$$

Proof. This is trivial when $q > x$ as we have the straightforward bound

$$T_f(\alpha) \ll x \sum_{k\le y} \frac{f'(k)}{f(k)} \ll x \log x.$$

So let $q \le x$. Using (51) with $q=1$, we obtain

$$\begin{aligned} T_f(\alpha) = T'_{f,1}(\alpha) &\ll \sum_{f(k)\le x^{1/2}} \min\left\{x\frac{f'(k)}{f(k)}, \frac{f'(k)}{\|f(k)\alpha\|}\right\} + \sum_{h\le x^{1/2}} \min\left\{\frac{x}{h}, \frac{1}{\|h\alpha\|}\right\} \\ &\ll \sum_{u\le x^{1/2}} \min\left\{\frac{x}{u}, \frac{1}{\|u\alpha\|}\right\}. \end{aligned}$$

An application of Lemma 5.3 yields

$$T_f(\alpha) \ll (xq^{-1}+x^{1/2}+q)\log x,$$

and the lemma follows immediately. □

We now have no problem to infer the required bound for $T_f(\alpha)$ on the minor arcs.

Proof of Lemma 4.1. Let $\alpha \in \mathfrak{m}_R$. Then we have either $\|q\alpha\| > R/x$ or $q > R$. By Dirichlet's theorem on diophantine approximation we can apply Lemma 5.4, and immediately obtain

$$\sup_{\mathfrak{m}_R} |T_f(\alpha)| \ll \frac{x}{R}\log x \ll x^{1/2+\varepsilon}.$$ □

It remains to prove the bounds for the integrals in Cor. 4.4. But first, we need to find bounds for $T'_{f,q}(\alpha)$ and $T''_{f,q}(\alpha)$ on the major arcs.

Lemma 5.5 ([Vau05, Lemma 2.11]) *Let* $q \in \mathbb{N}$ *and* $\alpha = a/q+\beta$ *with* $|\beta| \le 1/(2x^{1/2})$. *Then*

$$T'_{f,q}(\alpha) = T'_{f,q}(\beta) \ll \frac{x\log x}{q+qx|\beta|}.$$

Proof. The identity $T'_{f,q}(\alpha) = T'_{f,q}(\beta)$ is obvious from the properties of the exponential function. We first establish the equation

$$T'_{f,q}(\beta) \ll \sum_{u \le x^{1/2}/q} \frac{x}{qu + x\,\|qu\beta\|} \tag{52}$$

for an arbitrary β. From the definition (49) of $T'_{f,q}(\alpha)$ we find that

$$\begin{aligned} T'_{f,q}(\beta) &\ll \sum_{f(k) \le x^{1/2}/q} \min\left\{\frac{xf'(k)}{qf(k)}, \frac{f'(k)}{\|qf(k)\beta\|}\right\} + \sum_{h \le x^{1/2}/q} \min\left\{\frac{x}{qh}, \frac{1}{\|qh\beta\|}\right\} \\ &\ll \sum_{u \le x^{1/2}/q} \min\left\{\frac{x}{qu}, \frac{1}{\|qu\beta\|}\right\}. \end{aligned}$$

Note that for $\eta, \vartheta > 0$ we have

$$\min\{1/\eta, 1/\vartheta\} \le \frac{2}{\eta + \vartheta}.$$

This concludes the proof of (52).

In order to prove the lemma we note that for $u \le x^{1/2}/q$ and $|\beta| \le 1/(2x^{1/2})$, we have $|qu\beta| \le \frac{1}{2}$. We can thus replace $\|qu\beta\|$ by $|qu\beta|$ in (52), and immediately obtain

$$T'_{f,q}(\beta) \ll \frac{x}{q + qx|\beta|} \sum_{u \le x^{1/2}/q} \frac{1}{u} \ll \frac{x \log x}{q + qx|\beta|}.$$

□

We can give a similar bound for $T''_{f,q}(\alpha)$ on $\mathfrak{M}_R$.

Lemma 5.6 ([Vau05, Lemma 2.12]) *Let $(q, a) = 1$ and $|q\alpha - a| \le 1/(2x^{1/2})$. Then*

$$T''_{f,q}(\alpha) \ll \big(\min\{f(y), x^{1/2}\} + q\big) \log x.$$

Proof. Again, this is trivial for $q > x$. So let $q \le x$. By the definition (50) of $T''_{f,q}(\alpha)$, we then obtain

$$\begin{aligned} T''_{f,q}(\alpha) &\ll \sum_{\substack{f(k) \le \min\{f(y), x^{1/2}\} \\ q \nmid f(k)}} \frac{f'(k)}{\|f(k)\alpha\|} + \sum_{\substack{h \le \min\{f(y), x^{1/2}\} \\ q \nmid h}} \frac{1}{\|h\alpha\|} \\ &\ll \sum_{\substack{u \le \min\{f(y), x^{1/2}\} \\ q \nmid u}} \frac{1}{\|u\alpha\|}. \end{aligned} \tag{53}$$

We have the general inequality

$$\|u\alpha\| \ge \|ua/q\| - |u(\alpha - a/q)|.$$

By the definition (50) of $T''_{f,q}(\alpha)$, we have $q \neq u$, and so

$$\|ua/q\| \geq \frac{1}{q}.$$

Moreover, we have $u \leq x^{1/2}$ by (53) and $|q\alpha - a| \leq 1/(2x^{1/2})$ by assumption. Hence

$$|u(\alpha - a/q)| \leq 1/(2q).$$

Everything combined, we arrive at

$$T''_{f,q}(\alpha) \ll \sum_{\substack{u \leq \min\{f(y), x^{1/2}\} \\ q \nmid u}} \frac{1}{\|ua/q\|} \ll \left(\min\{f(y), x^{1/2}\}q^{-1} + 1\right) \sum_{r=1}^{q-1} \frac{1}{\|ra/q\|}.$$

On noting that

$$\sum_{r=1}^{q-1} \frac{1}{\|ra/q\|} \ll q \log x,$$

we conclude the proof. □

Finally, we are able to use Lemmata 4.2, 4.3, 5.5, and 5.6 to bound the error terms of $S_{\mathfrak{M}_R}(x, y)$.

Proof of Cor. 4.4. In light of equation (51) splitting $T_f(\alpha)$ into $T'_{f,q}(\alpha)$ and $T''_{f,q}(\alpha)$, we need to examine the four integrals

$$\int_{\mathfrak{M}_R} T'_{f,q}(\alpha)|\Delta(\alpha)|^2 \, \mathrm{d}\alpha, \qquad \int_{\mathfrak{M}_R} T''_{f,q}(\alpha)|\Delta(\alpha)|^2 \, \mathrm{d}\alpha,$$
$$\int_{\mathfrak{M}_R} T'_{f,q}(\alpha)|U^*(\alpha)\Delta(\alpha)| \, \mathrm{d}\alpha, \qquad \int_{\mathfrak{M}_R} T''_{f,q}(\alpha)|U^*(\alpha)\Delta(\alpha)| \, \mathrm{d}\alpha,$$

and establish that each of them is $O\left(x^{3/2+\varepsilon}\right)$.

We start by noting that for the choice $R = \frac{1}{2}x^{1/2}$ we can apply Lemma 5.6 to the major arcs, and obtain

$$\sup_{\alpha \in \mathfrak{M}_R} |T''_{f,q}(\alpha)| \ll x^{1/2} \log x.$$

Plugging now R into Lemmata 4.2 and 4.3 immediately establishes the desired bound in these cases.

The integrals containing $T'_{f,q}(\alpha)$ require a little more work as we need to perform a dyadic dissection. For this purpose define

$$\mathfrak{M}_0(1, 1) := \{\alpha \in \mathbb{R} : |\alpha - 1| \leq 1/x\}.$$

For $j \in \mathbb{N}$ and $q \leq 2^{j-1}$ define

$$\mathfrak{M}_j(q,a) := \left\{\alpha \in \mathbb{R} : 2^{j-1}/x < |q\alpha - a| \leq 2^j/x\right\},$$

and for $2^{j-1} < q \leq 2^j$ define

$$\mathfrak{M}_j(q,a) := \left\{\alpha \in \mathbb{R} : |q\alpha - a| \leq 2^j/x\right\}.$$

Now choose J such that $2^{J-1} < R \leq 2^J$. We chose this definition such that for any coprime a and q with $1 \leq a \leq q \leq R$, the union of the $\mathfrak{M}_j(q,a)$ with $0 \leq j \leq J$ and $q \leq 2^j$ contains $\mathfrak{M}_R(q,a)$ as defined in (27). We further define $\mathfrak{M}_j$ to be the union of the $\mathfrak{M}_j(q,a)$ with $1 \leq a \leq q \leq 2^j$ and $(a,q) = 1$. Then the union of the $\mathfrak{M}_j$ with $0 \leq j \leq J$ contains $\mathfrak{M}_R$.

We can now apply Lemma 5.5 to $\alpha \in \mathfrak{M}_j(q,a)$ with $1 \leq a \leq q \leq 2^j$ and $(a,q) = 1$, and obtain

$$T'_{f,q}(\alpha) \ll 2^{-j} x \log x. \tag{54}$$

An application of Lemma 4.2 yields

$$\sum_{q \leq 2^j} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}_j(q,a)} |\Delta(\alpha;q,a)|^2 \, \mathrm{d}\alpha \ll x^{\varepsilon} 2^{2j}.$$

Combining this with (54), we find that

$$\sum_{q \leq 2^j} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}_j(q,a)} T'_{f,q}(\alpha) |\Delta(\alpha;q,a)|^2 \, \mathrm{d}\alpha \ll x^{1+\varepsilon} 2^{j}.$$

We can now sum this result over all j with $0 \leq j \leq J$, and arrive at

$$\int_{\mathfrak{M}_R} T'_{f,q}(\alpha) |\Delta(\alpha;q,a)|^2 \, \mathrm{d}\alpha \ll x^{1+\varepsilon} \sum_{0 \leq j \leq J} 2^j \ll x^{1+\varepsilon} 2^J \ll x^{1+\varepsilon} R \ll x^{3/2+\varepsilon},$$

where we used that $2^{J-1} < R \leq 2^J$.

Similarly, we can apply Lemma 4.3 to $\mathfrak{M}_j(q,a)$, yielding

$$\sum_{q \leq 2^j} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}_j(q,a)} |U^*(\alpha;q,a)\Delta(\alpha;q,a)| \, \mathrm{d}\alpha \ll x^{1/2+\varepsilon} 2^{3j/4}.$$

Introducing the bound (54) again, we obtain

$$\sum_{q \leq 2^j} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}_j(q,a)} T'_{f,q}(\alpha) |U^*(\alpha;q,a)\Delta(\alpha;q,a)| \, \mathrm{d}\alpha \ll x^{3/2+\varepsilon} 2^{-j/4}.$$

We sum over all j, and conclude finally

$$\int_{\mathfrak{M}_R} T'_{f,q}(\alpha)|U^*(\alpha;q,a)\Delta(\alpha;q,a)|\,\mathrm{d}\alpha \ll x^{3/2+\varepsilon} \sum_{0\le j\le J} (2^{-1/4})^j \ll x^{3/2+\varepsilon}. \qquad \square$$

We can thus turn our attention to the analysis of $\Theta_f(H)$ at last.

6 Analysing Θ_f by Complex Analysis

We will deduce the asymptotic behaviour of $\Theta_f(H)$ by the means of the associated Dirichlet series $D(s)$ defined in (19). We will use a modified version of Perron's formula to obtain information about $\Theta_f(H)$ from $D(s)$.

Lemma 6.1 (Perron's Formula) *Let $a(n)$ be an arithmetic function, and*

$$L(a,s) = \sum_{n=1}^{\infty} a(n)n^{-s}$$

the associated Dirichlet series. If $L(a,s)$ converges absolutely for $\Re s > \sigma > 0$ and $x \in \mathbb{R}_{>0} \setminus \mathbb{N}$, then

$$\sum_{n \le x} a(n)(x-n)^2 = \frac{1}{\pi i} \int_{(\vartheta)} \frac{L(a,s)x^{s+2}}{s(s+1)(s+2)} \, \mathrm{d}s,$$

where $\vartheta > \sigma$ and the integration ranges from $\vartheta - i\infty$ to $\vartheta + i\infty$.

Proof. We will only sketch the proof as the details should be familiar. By absolute convergence of $L(a,s)$ we can exchange summation and integration, and obtain

$$\int_{(\vartheta)} \left(\sum_{n=1}^{\infty} a(n)n^{-s} \right) \frac{x^{s+2}}{s(s+1)(s+2)} \, \mathrm{d}s = \sum_{n=1}^{\infty} a(n)x^2 \int_{(\vartheta)} \left(\frac{x}{n}\right)^s \frac{\mathrm{d}s}{s(s+1)(s+2)}.$$

There are two cases to be checked. Let first $x/n < 1$. Consider integration along the rectangle with the corners $\vartheta \pm iT$ and $\vartheta + S \pm iT$, where $S, T > 0$. As the integrand has no poles inside this area, we conclude by the residue theorem

$$\int_{\square} \left(\frac{x}{n}\right)^s \frac{\mathrm{d}s}{s(s+1)(s+2)} = 0.$$

Since $x/n < 1$ the term $(x/n)^s$ will vanish as we let s tend to infinity. Thus we can extend the integration to infinity by letting S tend to infinity, where the integral from $\vartheta + S + iT$ to $\vartheta + S - iT$ will tend to zero. Likewise, we can extend T to infinity where the integrals from $\vartheta \pm iT$ to $\infty \pm iT$ will vanish. We obtain

$$\int_{(\vartheta)} \left(\frac{x}{n}\right)^s \frac{\mathrm{d}s}{s(s+1)(s+2)} = 0$$

for $n > x$.

Let now $x/n > 1$. Here, we consider integration along the rectangle with the corners $\vartheta \pm iT$ and $\vartheta - S \pm iT$, where $S, T > 0$. For S large enough, the integrand has single poles at $s = 0$, $s = -1$, and $s = -2$ inside this area. Hence

$$\frac{1}{2\pi i} \int_{\square} \left(\frac{x}{n}\right)^s \frac{\mathrm{d}s}{s(s+1)(s+2)} = R(0) + R(-1) + R(-2),$$

where $R(s)$ denotes the residue of the integrand at s. We calculate

$$R(0) = \frac{1}{2}, \qquad R(-1) = -\frac{n}{x}, \qquad R(-2) = \frac{n^2}{2x^2}.$$

With the same argument as above we let S and T tend to infinity, and conclude

$$\frac{1}{\pi i} \int_{(\vartheta)} \left(\frac{x}{n}\right)^s \frac{\mathrm{d}s}{s(s+1)(s+2)} = \left(1 - \frac{n}{x}\right)^2$$

for $n < x$. Assembling these results proves the claimed identity. Note that we excluded the case $x \in \mathbb{N}$ for technical convenience. In fact, we could obtain a similar formula for $x \in \mathbb{N}$ where the last term in the sum over n needs to be modified. However, as we are only interested in the asymptotic behaviour, we chose not to include this case here. □

Write now for short

$$I_H(s) = \frac{D(s) H^{s+2}}{s(s+1)(s+2)}. \tag{55}$$

Note that $D(s)$ is the Dirichlet series associated to $W(n)/n$, so we can apply Lemma 6.1, and gain immediately that

$$\Theta_f(H) = \frac{1}{\pi i} \int_{(2)} I_H(s)\, \mathrm{d}s, \tag{56}$$

where the line of integration could be along any $\vartheta > 0$. This equation constitutes our interest in $D(s)$ – we will later gain an asymptotic for $\Theta_f(H)$ through the residues of $I_H(s)$. In order to calculate these, we need some relations connecting $D(s)$ to $\zeta(s)$ which will also prove the meromorphic behaviour claimed in Prop. 2.4. Let us first take a look at $W(h)$. By Lemma 4.7, $w_h(q)$ is multiplicative as a function of q. Hence $W(h)$ can be expanded as an Euler product of the form

$$\begin{aligned} W(h) &= \prod_p \sum_{l=0}^{\infty} G(p^l)^2 w_h(p^l) \\ &= \prod_p \left(1 + \frac{w_h(p) + w_h(p^2)}{(p^2-1)^2}\right) \\ &= \prod_{p \nmid h} P_0^{-1} \cdot \prod_{p \| h} P_1 \cdot \prod_{p^2 \mid h} P_2, \end{aligned}$$

where

$$P_0(p) = P_0 = \left(1 + \frac{\varrho(p^2) - 1}{(p^2-1)^2}\right)^{-1} = 1 - \frac{\varrho(p^2) - 1}{p^4 - 2p^2 + \varrho(p^2)} = 1 + O\left(p^{-4}\right),$$
$$P_1(p) = P_1 = 1 + \frac{p\varrho(p) - 1}{(p^2-1)^2} = 1 + O\left(p^{-3}\right),$$

and

$$P_2(p) = P_2 = 1 + \frac{1}{p^2 - 1} = 1 + O\left(p^{-2}\right).$$

Note that we can give these bounds since $\varrho(p^l) \leq d^l$ for primes p. From the Euler product we deduce that $W(h)/W(1)$ is multiplicative with

$$\frac{W(h)}{W(1)} = \prod_{p \| h} P_0 P_1 \cdot \prod_{p^2 | h} P_0 P_2.$$

In turn, we can find that

$$\frac{D(s)}{W(1)} = \sum_{n=1}^{\infty} \frac{W(n)}{W(1)} n^{-s-1} = \prod_p D_p(s),$$

where the Euler factor is

$$D_p(s) = 1 + P_0 P_1 p^{-s-1} + P_0 P_2 \frac{p^{-2s-2}}{1 - p^{-s-1}}.$$

Writing $\xi = p^{-s-1}$ we can express the Euler factor as

$$D_p(s) = 1 + P_0 P_1 \xi + P_0 P_2 \frac{\xi^2}{1 - \xi}.$$

With these means we find the relations

$$\begin{aligned}(1-\xi)D_p(s) &= 1 + (P_0P_1 - 1)\xi + P_0(P_2 - P_1)\xi^2, \\ \left(1 - \xi^2/p^2\right)(1-\xi)D_p(s) &= 1 + (P_0P_1 - 1)\xi + (p^2P_0(P_2 - P_1) - 1)\frac{\xi^2}{p^2} \\ &\quad - (P_0P_1 - 1)\frac{\xi^3}{p^2} - P_0(P_2 - P_1)\frac{\xi^4}{p^2},\end{aligned}$$

and

$$\frac{(1-\xi^2/p^2)(1-\xi)}{1 - \xi^4/p^4} D_p(s) = 1 + \frac{\xi(P_0P_1 - 1)}{1 + \xi^2/p^2} + \frac{\xi^2/p^2}{1 + \xi^2/p^2}\left(p^2P_0(P_2 - P_1) - 1\right).$$

This motivates the definitions

$$E_1(s) := \prod_p \left(1 + (P_0P_1 - 1)p^{-s-1} + P_0(P_2 - P_1)p^{-2s-2}\right),$$

$$E_2(s) := \prod_p \left(1 + (P_0P_1 - 1)p^{-s-1} + (p^2P_0(P_2 - P_1) - 1)p^{-2s-4} \right.$$
$$\left. -(P_0P_1 - 1)p^{-3s-5} - P_0(P_2 - P_1)p^{-4s-6}\right),$$

and

$$E_3(s) := \prod_p \left(1 + \frac{p^{s+3}(P_0P_1 - 1)}{p^{2s+4} + 1} + \frac{p^2P_0(P_2 - P_1) - 1}{p^{2s+4} + 1}\right).$$

We obtain the functional equations

$$\frac{D(s)}{W(1)} = \begin{cases} \zeta(s+1)E_1(s), \\ \zeta(s+1)\zeta(2s+4)E_2(s), \\ \frac{\zeta(s+1)\zeta(2s+4)}{\zeta(4s+8)}E_3(s). \end{cases} \tag{57}$$

A closer look at P_0, P_1, and P_2 reveals that

$$P_0P_1 = \frac{p^4 - 2p^2 + p\varrho(p)}{p^4 - 2p^2 + \varrho(p^2)} = 1 + \frac{p\varrho(p) - \varrho(p^2)}{p^4 - 2p^2 + \varrho(p^2)} = 1 + O\left(p^{-3}\right),$$
$$P_0(P_2 - P_1) = \frac{p^2 - p\varrho(p)}{p^4 - 2p^2 + \varrho(p^2)} = O\left(p^{-2}\right),$$

and

$$p^2P_0(P_2 - P_1) - 1 = \frac{-p^3\varrho(p) + 2p^2 + \varrho(p^2)}{p^4 - 2p^2 + \varrho(p^2)} = O\left(p^{-1}\right).$$

Hence

$$E_1(s) := \prod_p \left(1 + O\left(p^{-s-4}\right) + O\left(p^{-2s-4}\right)\right),$$
$$E_2(s) := \prod_p \left(1 + O\left(p^{-s-4}\right) + O\left(p^{-2s-5}\right) + O\left(p^{-3s-8}\right) - O\left(p^{-4s-8}\right)\right),$$

and

$$E_3(s) := \prod_p \left(1 + O\left(p^{-s-4}\right) + O\left(p^{-2s-5}\right)\right),$$

and so $E_1(s)$, $E_2(s)$, and $E_3(s)$ converge absolutely and are analytic in the half-plane

$\Re s > -\frac{3}{2}$, $\Re s > -\frac{7}{4}$, and $\Re s > -2$, respectively. The relations (57) yield thus a meromorphic continuation of $D(s)$ to the half plane $\Re s > 2$. In particular, $D(s)$ is meromorphic on the half-plane $\Re s \geq -\frac{7}{4}$ with the only single poles at $s = 0$ and $s = -\frac{3}{2}$, proving Prop. 2.4.

We can now go back to (56). By Prop. 2.4 we can move the line of integration to $\vartheta = -\frac{7}{4}$, picking up the residues of $I_H(s)$ at $s = 0$, $s = -1$, and $s = -\frac{3}{2}$. Writing $R(0)$, $R(-1)$, and $R(-3/2)$ for the residues at $s = 0$, $s = -1$, and $s = -\frac{3}{2}$, respectively, we can conclude by the residue theorem that

$$\frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT} + \int_{2+iT}^{-\frac{7}{4}+iT} + \int_{-\frac{7}{4}+iT}^{-\frac{7}{4}-iT} + \int_{-\frac{7}{4}-iT}^{2-iT}\right) I_H(s)\,\mathrm{d}s = R(0) + R(-1) + R(-3/2).$$

Invoking standard estimates (cf. [Tit86, Ch. V]) on the ζ-function in the functional equations (57), we can extend T to infinity. In light of (56), this yields

$$\Theta_f(H) = 2R(0) + 2R(-1) + 2R(-3/2) + \frac{1}{\pi i}\int_{(-7/4)} I_H(s)\,\mathrm{d}s. \tag{58}$$

Another standard estimates gives us as a bound for the integral

$$\int_{(-7/4)} I_H(s)\,\mathrm{d}s \ll H^{1/4}.$$

We thus need the values of the residues of $I_H(s)$ for an asymptotic of $\Theta_f(H)$.

Lemma 6.2 *With notation as above we have for the residues of $I_H(s)$:*

(i) $R(0) = \frac{1}{2}\Gamma_0 H^2 \log H + \Gamma_0' H^2$, *where*

$$\Gamma_0 = W(1)E_1(0)$$

and

$$\Gamma_0' = \frac{1}{2}\gamma W(1)E_1(0) + \frac{1}{2}W(1)E_1'(0) - \frac{3}{4}W(1)E_1(0).$$

(ii) $R(-1) = \frac{1}{2}\zeta(2)H$.

(iii) $R(-3/2) = \Gamma_{-3/2}H^{1/2}$, *where*

$$\Gamma_{-3/2} = \frac{8}{3}\zeta(-1/2)W(1)E_2(-3/2).$$

Proof. (*i*) We will need the expansion (cf. [Tit86, (2.1.16)]),

$$\frac{\zeta(s+1)}{s} = \frac{1}{s^2} + \frac{\gamma}{s} + O(1), \tag{59}$$

where γ is the Euler–Mascheroni constant. From Prop. 2.4 we know that $D(s)$ has a single pole at $s = 0$. This implies that $I_H(s)$ has a double pole there. Hence

$$R(0) = \lim_{s \to 0} \frac{\mathrm{d}}{\mathrm{d}s} \left(s^2 I_H(s) \right).$$

Using the first equation in (57) together with the expansion (59) we obtain

$$s^2 I_H(s) = \left(1 + \gamma s + O\left(s^2\right)\right) \frac{W(1)E_1(s)H^{s+2}}{(s+1)(s+2)}.$$

The derivative with respect to s is then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \left(s^2 I_H(s) \right) &= \left(\gamma + O(s)\right) W(1)E_1(s)H^{s+2}(s+1)^{-1}(s+2)^{-1} \\ &\quad + \left(1 + \gamma s + O\left(s^2\right)\right) W(1)H^{s+2}(s+1)^{-2}(s+2)^{-2} \\ &\quad \cdot \left(\left(E_1'(s) + E_1(s)\log H\right)(s+1)(s+2) - E_1(s)(2s+3) \right). \end{aligned}$$

We conclude

$$R(0) = \frac{1}{2}\Gamma_0 H^2 \log H + \Gamma_0' H^2,$$

where

$$\Gamma_0 = W(1)E_1(0),$$

and

$$\Gamma_0' = \frac{1}{2}\gamma W(1)E_1(0) + \frac{1}{2}W(1)E_1'(0) - \frac{3}{4}W(1)E_1(0).$$

(*ii*) As $D(s)$ is analytic in a neighbourhood of $s = -1$, we find that

$$R(-1) = \lim_{s \to -1} (s+1)I_H(s).$$

Using again the first relation in (57) this yields

$$(s+1)I_H(s) = \frac{W(1)\zeta(s+1)E_1(s)H^{s+2}}{s(s+2)}.$$

Noting that $\zeta(0) = -\frac{1}{2}$ we arrive at

$$R(-1) = \frac{1}{2}\Gamma_{-1}H,$$

where

$$\begin{aligned}\Gamma_{-1} &= W(1)E_1(-1)\\ &= \prod_p P_0^{-1}\left(1 + (P_0P_1 - 1) + P_0(P_2 - P_1)\right)\\ &= \prod_p P_2 = \prod_p \frac{1}{1 - p^{-2}} = \zeta(2).\end{aligned}$$

(*iii*) Again, $I_H(s)$ has a single pole at $s = -\frac{3}{2}$. Thus, using the second equation in (57) together with the fact that

$$\lim_{s \to -\frac{3}{2}} \left(s + \frac{3}{2}\right)\zeta(2s + 4) = 1,$$

we conclude

$$R(-3/2) = \lim_{s \to -\frac{3}{2}} \left(s + \frac{3}{2}\right) I_H(s) = \Gamma_{-3/2}H^{1/2},$$

where

$$\Gamma_{-3/2} = \frac{8}{3}\zeta(-1/2)W(1)E_2(-3/2).$$

□

Before we proceed, we will take a closer look to the constant Γ_0. Writing out the definitions of $W(1)$ and $E_1(0)$, and sorting according to prime powers yields

$$\begin{aligned}\Gamma_0 &= \prod_p \frac{p^6 - 2p^4 + p^2 + p^2\varrho(p) - p\varrho(p) + p^2\varrho(p^2) - p\varrho(p^2)}{p^2(p^2 - 1)^2}\\ &= \prod_p \left(1 + \frac{\varrho(p)(p-1)}{p(p^2 - 1)^2} + \frac{\varrho(p^2)(p^2 - p)}{p^2(p^2 - 1)^2}\right).\end{aligned}$$

We can then transform the product back into the sum

$$\Gamma_0 = \sum_{n=1}^{\infty} \varphi(n)G(n)^2\frac{\varrho(n)}{n}. \qquad (60)$$

In this form, the constant will reappear in Chapter 7.

All we need to do now is to plug the values of the residues provided by Lemma 6.2 into equation (58). This yields the asymptotic

$$\Theta_f(H) = \Gamma_0 H^2 \log H + 2\Gamma_0' H^2 + \zeta(2)H + 2\Gamma_{-3/2} H^{1/2} + O\left(H^{1/4}\right).$$

Combining this with Prop. 2.3 concludes the proof of Cor. 2.5, where

$$C_f = -2\zeta(2)^{-2}\Gamma_{-3/2} = -\frac{16\zeta(-1/2)}{3\zeta(2)^2} W(1) E_2(-3/2). \tag{61}$$

Note that

$$\zeta(-1/2) = -0.207886224977355\ldots$$

and thus the numerical value of the constant is positive.

What remains now is to analyse $\Phi_f(y)$.

7 Analysing Φ_f by Multiplicative Number Theory

The final step is to analyse the asymptotic behaviour of $\Phi_f(y)$. As it turns out, this depends crucially on the constant Γ_0 in the form of (60). But first, we need to rearrange the innermost sum of $\Phi_f(y)$ in (14) to connect it to the function $G(n)$ as defined in (15).

Lemma 7.1 ([Vau98b, Lemma 2.4]) *Let $k \in \mathbb{N}$. Then*

$$\sum_{r|k} g(k,r)^2 \varphi(k/r) = \zeta(2)^{-2} k^{-1} \sum_{r|k} \varphi(r) G(r)^2.$$

Proof. We will evaluate

$$\lambda = \lim_{x\to\infty} x^{-2} \sum_{l=1}^{k} \left| \sum_{n\le x} \mu(n)^2 e(ln/k) \right|^2$$

in two ways. First, we exploit orthogonality of the additive characters, and obtain

$$\begin{aligned} \sum_{l=1}^{k} \left| \sum_{n\le x} \mu(n)^2 e(ln/k) \right|^2 &= k \sum_{n\le x} \sum_{\substack{m\le x \\ n\equiv m\ (k)}} \mu(n)^2 \mu(m)^2 \\ &= k \sum_{l=1}^{k} \left(\sum_{\substack{n\le x \\ n\equiv l\ (k)}} \mu(n)^2 \right)^2 \\ &= k \sum_{l=1}^{k} Q(x;k,l)^2. \end{aligned}$$

Thus, with (2) and Lemma 3.3,

$$\lambda = k \sum_{l=1}^{k} \lim_{x\to\infty} \frac{Q(x;k,l)^2}{x^2} = k \sum_{l=1}^{k} g(k,l)^2 = k \sum_{r|k} g(k,r)^2 \varphi(k/r).$$

On the other hand, by sorting according to the divisors of k, we have

$$\sum_{l=1}^{k} \left| \sum_{n\le x} \mu(n)^2 e(ln/k) \right|^2 = \sum_{r|k} \sum_{\substack{b=1 \\ (b,r)=1}}^{r} \left| \sum_{n\le x} \mu(n)^2 e(bn/r) \right|^2 = \sum_{r|k} \sum_{\substack{b=1 \\ (b,r)=1}}^{r} |U(b/r)|^2.$$

We now need

$$\lim_{x\to\infty} x^{-1} U(b/r) = \nu(r) \tag{62}$$

for coprime b and r. So let us write out the definition (26) of $U(\alpha)$ and sort according to the residue classes of n modulo r:

$$U(b/r) = \sum_{a=1}^{r} \sum_{\substack{n \le x \\ n \equiv a \ (r)}} \mu(n)^2 e(bn/r) = \sum_{a=1}^{r} e(ab/r) Q(x; r, a).$$

Using once more (2) we obtain

$$\lim_{x\to\infty} x^{-1} U(b/r) = \sum_{a=1}^{r} e(ab/r) \lim_{x\to\infty} \frac{Q(x;r,a)}{x} = \sum_{a=1}^{r} g(r,a) e(ab/r).$$

Since we assumed b and r to be coprime both a and ab run through a complete coset representative modulo r. Moreover, by Lemma 3.1 the function $g(r,a)$ depends only on the gcd of r and a. Hence, with the definition (31) of $\nu(r)$,

$$\lim_{x\to\infty} x^{-1} U(b/r) = \sum_{a=1}^{r} g(r,a) e(a/r) = \nu(r),$$

confirming (62). Thus

$$\lambda = \sum_{r|k} \sum_{\substack{b=1 \\ (b,r)=1}}^{r} \lim_{x\to\infty} \frac{|U(b/r)|^2}{x^2} = \sum_{r|k} |\nu(r)|^2 \varphi(r).$$

The claimed identity then follows directly from Lemma 4.5. □

Plugging now Lemma 7.1 into the definition (14) of $\Phi_f(y)$ yields

$$\Phi_f(y) = \zeta(2)^{-2} \sum_{y_1 < k \le y} \frac{f'(k)}{f(k)} \sum_{r|f(k)} \varphi(r) G(r)^2.$$

Define temporarily

$$\Phi'_f(y) = \sum_{k \le y} \frac{f'(k)}{f(k)} \sum_{r|f(k)} \varphi(r) G(r)^2. \tag{63}$$

We will handle the factor $f'(k)/f(k)$ by summation by parts. Hence we first need to examine the sum without this factor.

Lemma 7.2 *Let $k \in \mathbb{N}$. Then*

$$\sum_{k \le y} \sum_{r|f(k)} \varphi(r) G(r)^2 = \Gamma_0 y + \widetilde{c}_f + O\left(y^{-2d+1}\right),$$

where $\widetilde{c}_f$ is a constant depending on f only.

Proof. We first exchange the order of summation and obtain

$$\begin{aligned}\sum_{k\le y}\sum_{r\mid f(k)}\varphi(r)G(r)^2 &= \sum_{r\le f(y)}\varphi(r)G(r)^2 \sum_{\substack{k\le y\\ f(k)\equiv 0\ (r)}} 1\\ &= \sum_{r\le f(y)}\varphi(r)G(r)^2\varrho(r)\left(\frac{y}{r}+O\left(1\right)\right).\end{aligned}$$

The error here is bounded by

$$\ll \sum_{r\le f(y)}\varphi(r)G(r)^2\varrho(r) = \widetilde{c}_f + O\left(f(y)^{-2}\right).$$

We extend the summation over r to infinity, and obtain

$$\sum_{r\le f(y)}\varphi(r)G(r)^2\frac{\varrho(r)}{r} = \Gamma_0 + O\left(f(y)^{-2}\right),$$

where Γ_0 is the constant in the shape of (60). Hence

$$\begin{aligned}\sum_{k\le y}\sum_{r\mid f(k)}\varphi(r)G(r)^2 &= y\sum_{r=1}^{\infty}\varphi(r)G(r)^2\frac{\varrho(r)}{r} + \widetilde{c}_f + O\left(y^{-2d+1}\right)\\ &= \Gamma_0 y + \widetilde{c}_f + O\left(y^{-2d+1}\right).\end{aligned}$$

□

We now write

$$F(t) = \frac{f'(t)}{f(t)}.$$

Note that $F(t) \ll t^{-1}$ and that the antiderivative of $F(t)$ is $\log f(t)$. Summation by parts combined with Lemma 7.2 then yields

$$\begin{aligned}\Phi_f'(y) &= \Gamma_0 yF(y) + \widetilde{c}_f F(y) - \Gamma_0\int_1^y tF'(t)\,\mathrm{d}t - \widetilde{c}_f\int_1^y F'(t)\,\mathrm{d}t + O\left(y^{-2d+1}\right)\\ &= \Gamma_0 yF(y) - \Gamma_0\int_1^y tF'(t)\,\mathrm{d}t + \widetilde{c}_f F(1) + O\left(y^{-2d+1}\right).\end{aligned}$$

We can now evaluate the integral by integration by parts. Hence

$$\int_1^y tF'(t)\,\mathrm{d}t = yF(y) - \log f(y) + \log f(1) - F(1).$$

These results combined yield

$$\Phi_f'(y) = \Gamma_0\log f(y) + c_f + O\left(y^{-2d+1}\right),$$

where

$$c_f = \Gamma_0 \left(\frac{f'(1)}{f(1)} - \log f(1) \right) + \widetilde{c}_f \frac{f'(1)}{f(1)}.$$

Hence

$$\Phi_f(y) = \zeta(2)^{-2} \left(\Phi'_f(y) - \Phi'_f(y_1) \right) = \zeta(2)^{-2} \Gamma_0 \log \left(f(y)/f(y_1) \right) + O\left(y^{-2d+1} \right),$$

proving Prop. 2.6, and thus completing the final step in the proof of our main result Thm. 2.1.

8 Conclusions and Outlook

The methods applied in this survey are standard in analytic number theory therefore we can expect our main result Thm. 2.1 to be generalisable in a number of ways. The most natural extension would be to k-free numbers. Considering the work of R. C. Vaughan [Vau05], most of the adjustments would be straightforward. The equivalent of the function $G(n)$ would only vanish for prime powers p^t with $t > k$ though, leading to more complex expressions for $w_h(q)$ than ours in Lemma 4.7, and thus to more cases to be distinguished between in the product of $W(h)$.

One could also think of applying the methods presented to more general sequences as Vaughan has done in his papers [Vau98a, Vau98b], following the work of C. Hooley [Hoo75b]. Indeed, the majority of the techniques relies on the general behaviour of the asymptotic density $g(k,l)$ rather than on special properties of the squarefree numbers. Thus our result could be transferred to any sequence whose asymptotic density is positive and exhibits similar behaviour to $g(k,l)$.

It is also worth thinking of more general functions f. Although we used the fact that f is a polynomial in a number of places, one may substitute it by a function of more rapid growth. J. Brüdern and A. Perelli [BP98] have examined functions of the shape

$$f(k) = \left\lfloor \exp\left((\log k)^{\vartheta}\right)\right\rfloor,$$

where the exponent ϑ is a positive number with $\vartheta < \frac{3}{2}$. It should be possible to adjust their methods to our variance in order to obtain an asymptotic formula for this case as well.

As mentioned in the introduction, the gained error bounds are not the sharpest possible. Following Vaughan's more elaborate examinations, the error bound $O\left(x^{3/2+\varepsilon}\right)$ may be replaced by a term of the form $O\left(x^{3/2-\delta}\right)$. However, as our focus was on the main term, we chose to avoid these technicalities.

Also, we have not paid too much attention on small values for $f(y)$, where better error bounds than $O\left(x^2\right)$ can be expected. Again, in the range $0 \leq f(y) \ll x^{2/3+\varepsilon}$ no asymptotic formula can be obtained, which is why we omitted a further discussion. For similar reasons, we did not include the case $f(y) > x$.

A different approach would be to take a step away from variances, and instead examine an analogue to the Bombieri–Vinogradov theorem. H. Mikawa and T. P. Peneva [MP05] found an upper bound for primes in spaced moduli, and combining their work with the methods presented here it should be possible to find a similar estimate for

$$\sum_{k \leq y} f'(k) \max_{1 \leq l \leq f(k)} |E(x; f(k), l)|,$$

where $f(y) \leq x^{\vartheta}$, and ϑ is an exponent to be chosen as large as possible. Note that

R. C. Orr [Orr71] proved in the non-restricted case that

$$\sum_{k\leq y} \max_{1\leq l\leq k} |E(x;k,l)| \ll x(\log x)^{-A},$$

where $y \leq x^{2/3}(\log x)^{-A-1}$, and $A > 0$ is arbitrary. The exponent ϑ can therefore not expected to be extended beyond $\frac{2}{3}$.

However, even with these generalisations in mind, the required methods would still include the Hardy–Littlewood circle method and the analysis of Dirichlet series. Combining these classic techniques with new ideas, analytic number theory remains an active and vivid field of research.

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Notations

Throughout the paper, constants c_1, c_2, $\dots$, are assumed to be positive. The symbol ε denotes a (sufficiently small) positive number, not necessarily with the same value in every occurrence. Implied constants may depend on f and ε, but not on x or y.

$a \mid b$ the integer a is a divisor of the integer b

$p^t \parallel b$ the prime power p^t divides the integer b, but p^{t+1} does not

$\lfloor \alpha \rfloor$ largest integer not exceeding α

(a, b) greatest common divisor of the integers a and b

$\int_{(\vartheta)}$ short for $\lim_{T\to\infty} \int_{\vartheta-iT}^{\vartheta+iT}$

$\ll, \gg$ the expression $g(x) \ll f(x)$ is short for $g(x) = O(f(x))$, and $f(x) \gg g(x)$ means $g(x) \ll f(x)$

$||\alpha||$ distance to the closest integer

$\mathbb{C}$ the complex numbers

C_f constant of the main term depending on f, an expression is given in (61)

$c_q(n)$ Ramanujan's sum

d degree of the polynomial f

$\Delta(\alpha; q, a)$ difference between $U(\alpha)$ and $U^*(\alpha; q, a)$, defined in (33)

$D(s)$ Dirichlet series associated to $W(h)$, defined in (19)

$E_1(s), E_2(s), E_3(s)$ analytic functions yielding functional equations for $D(s)$

$e(\alpha)$ modified exponential function $e(\alpha) := \exp(2\pi i\alpha)$

$E(x; k, l)$ the error term $Q(x; k, l) - g(k, l)x$, defined in (3)

f a fixed, integer-valued polynomial of degree d

γ Euler–Mascheroni constant $\gamma = 0.5772156649\dots$

$\Gamma_0, \Gamma_0', \Gamma_{-3/2}$ constants that appear in the residues of $I_H(s)$

$g(k, l)$ the asymptotic density of the squarefree numbers, defined in (2)

$G(n)$ multiplicative function defined in (15)

$I_H(s)$ integrand containing $D(s)$ tailored at Perron's formula, defined in (55)

$\Im(s)$ imaginary part of the complex number s

$J(\alpha)$ sum over $e(n\alpha)$, defined in (30)

$M_0(x, y)$ reduction of $S_{\mathfrak{M}_R}(x, y)$, defined in (36)

$\mathfrak{M}_R$ the major arcs depending on the parameter R

$\mathfrak{m}_R$ the minor arcs depending on the parameter R

$\mu(n)$ the Möbius function, its square is the indicator function of the squarefree numbers

$\mathbb{N}$ the positive integers

$\nu(q)$ multiplicative function closely related to $G(q)$, defined in (31)

$O(f(x))$ a (positive) function $g(x)$ is $O(f(x))$ if $|f(x)|$ is asymptotically dominant, i. e., if $g(x) \leq C|f(x)|$ for some constant $C > 0$

$o(f(x))$ a (positive) function $g(x)$ is $o(f(x))$ if $|f(x)|$ grows faster than $g(x)$, i. e., if $\lim g(x)/|f(x)| = 0$

p a prime number

P_0, P_1, P_2 factors in the Euler product of $W(h)$ depending on p

$\varphi(n)$ Euler's totient function

$\Phi_f(y)$ sum containing $g(k, l)$, defined in (14)

$\Phi'_f(y)$ completed version of $\Phi_f(y)$, defined in (63)

$\mathbb{Q}$ the rational numbers

$Q(x; k, l)$ the number of squarefree integers congruent to l modulo k not exceeding x, defined in (1)

R parameter of the Farey dissection, here $R = \frac{1}{2}x^{1/2}$

$\mathbb{R}$ the real numbers

$\Re(s)$ real part of the complex number s

$\varrho(m)$ multiplicative function, number of solutions of the congruence $f(a) \equiv 0$ (m) with $0 \leq a < m$

$S_0(x, y)$ sum containing the main contribution in $V_f(x, y)$, defined in (13)

$\sigma_0(n)$ the divisor function

$S_{\mathfrak{M}_R}(x,y)$ the restriction of $S_0(x,y)$ to the major arcs, defined in (29)

$T_f(\alpha)$ sum over $e(\alpha h f(k))$, defined in (25)

$T'_{f,q}(\alpha)$ modified version of $T_f(\alpha)$ where the outer summation variable is divisible by q, defined in (49)

$T''_{f,q}(\alpha)$ modified version of $T_f(\alpha)$ where the outer summation variable is not divisible by q, defined in (50)

$\Theta_f(H)$ sum accessible by the Dirichlet series $D(s)$, defined in (18)

$U(\alpha)$ sum over $\mu(n)^2 e(\alpha n)$, defined in (26)

$U^*(\alpha; q, a)$ simplified version of $U(\alpha)$, defined in (32)

$V'_f(x,y)$ truncated version of $V_f(x,y)$

$V_f(x,y)$ the square mean of the error term over the residue classes, defined in (12)

$W(h)$ series combining $G(q)$ and $w_h(q)$, defined in (17)

$w_h(q)$ multiplicative function of q, defined in (16)

x a (sufficiently large) fixed real number

y a (sufficiently large) fixed real number such that $f(y) \leq x$

y_1 unique $y > 0$ such that $f(y) = x^{1/4}$

$y(h)$ defined by means of the equation $f(y(h)) = \min\{f(y), x/h\}$

$\mathbb{Z}$ the integers

$\zeta(s)$ Riemann's ζ-function

Plagiatserklärung

Hiermit versichere ich, dass ich die Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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Cambridge, May 7, 2012 ______________________________